On Description of Isomorphism Classes of filiform Leibniz algebras in dimensions 7 and 8

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Abstract

The paper concerns the classification problem of a subclass of complex filiform Leibniz algebras in dimensions 7 and 8. This subclass arises from naturally graded filiform Lie algebras. We give a complete list of isomorphism classes of algebras including Lie case. In parametric families cases, the corresponding orbit functions (invariants) are given. In discrete orbits case, we show representatives of the orbits.

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1 Introduction

Leibniz algebras were introduced by J. -L. Loday [5],[6]. A skew-symmetric Leibniz algebra is a Lie algebra. The main motivation of J. -L. Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity in algebraic $K-$theory. Besides this purely algebraic motivation, some relationships with classical geometry, non-commutative geometry and physics have been recently discovered. The present paper deals with the low-dimensional case of subclass of filiform Leibniz algebras. This subclass, arises from naturally graded filiform Lie algebras.

Definition 1.1. *An algebra L over a field K is called a Leibniz algebra, if its bilinear operation $[\cdot,\cdot]$ satisfies the following Leibniz identity:* $[x,[y,z]] = [[x,y],z] - [[x,z],y],$ *for any* $x,y,z \in L$.

Onward, all algebras are assumed to be over the fields of complex numbers $\mathbb{C}$.

Let L be a Leibniz algebra. We put:

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1.$$

Definition 1.2. *A Leibniz algebra L is said to be nilpotent, if there exists $s \in \mathbb{N}$, such that*

$$L^1 \supset L^2 \supset ... \supset L^s = \{0\}.$$

Definition 1.3. *A Leibniz algebra L is said to be filiform, if $dimL^i = n - i$, where $n = dimL$, and $2 \leq i \leq n$.*

We denote by $Leib_n$ the set of all $n-$dimensional filiform Leibniz algebras.

The following theorem from [4] splits the set of fixed dimension filiform Liebniz algebras in to three disjoint subsets.

Theorem 1.1. *Any $(n+1)-$dimensional complex filiform Leibniz algebra L admits a basis $\{e_0, e_1, ..., e_n\}$ called adapted, such that the table of multiplication of L has one of the following forms, where non defined products are zero:*

$$FLeib_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + ... + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + ... + \alpha_{n+1-j} e_n, & 1 \leq j \leq n-2, \\ \quad \alpha_3, \alpha_4, ..., \alpha_n, \theta \in \mathbb{C}. & \end{cases}$$

$$SLeib_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + ... + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + ... + \beta_{n+1-j} e_n, & 2 \le j \le n-2, \\ \qquad \beta_3, \beta_4, ..., \beta_n, \gamma \in \mathbb{C}. & \end{cases}$$

$$TLeib_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_0] = b_{0,0} e_n, & \\ [e_0, e_1] = -e_2 + b_{0,1} e_n, & \\ [e_1, e_1] = b_{1,1} e_n, & \\ [e_i, e_j] = a^1_{i,j} e_{i+j+1} + \cdots + a^{n-(i+j+1)}_{i,j} e_{n-1} + b_{i,j} e_n, & 1 \le i < j \le n-1, \\ [e_i, e_j] = -[e_j, e_i], & 1 \le i < j \le n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i} e_n, & \\ \text{where } a^k_{i,j}, b_{i,j} \in \mathbb{C},\ b_{i,n-i} = b \text{ whenever } 1 \le i \le n-1, \text{ and } b = 0 \text{ for even } n. & \end{cases}$$

It should be mentioned that in the theorem above the structure constants $\alpha_3, \alpha_4, ..., \alpha_n, \theta$ and $\beta_3, \beta_4, ..., \beta_n, \gamma$ in the first two cases are free, however, in the third case there are relations among $a^k_{i,j}, b_{i,j}$ that are different in each fixed dimension (see Lemma 2.1 and Lemma 2.2). According to the theorem each class has the so-called adapted base change sending an adapted basis to adapted and they can be studied autonomously. The classes $FLeib_n, SLeib_n$ in low dimensional cases, have been considered in [8], [9]. The general methods of classification for $Leib_n$ has been given in [1], [2] and [7]. This paper deals with the classification problem of low-dimensional cases of $TLeib_n$.

Note that the class $TLeib_n$ contains all $n-$dimensional filiform Lie algebras.

The outline of the paper is as follows. Section 1 is an introduction to a subclass of Leibniz algebras that we are going to investigate. Section 2 presents the main results of the paper consisting of a complete classification of a subclass of low dimensional filiform Leibniz algebras. Here, for 7- and 8-dimensional cases we give complete classification. For parametric family cases the corresponding invariant functions are presented.

Definition 1.4. *Let $\{e_0, e_1, ..., e_n\}$ be an adapted basis of L from $TLeib_{n+1}$. Then a nonsingular linear transformation $f : L \to L$ is said to be adapted if the basis $\{f(e_0), f(e_1), ..., f(e_n)\}$ is adapted.*

The subgroup of GL_{n+1} consisting of all adapted transformations is denoted by G_{ad}. The following proposition specifies elements of G_{ad}.

Proposition 1.1. *Any adapted transformation f in $TLeib_{n+1}$ can be represented as follows:*

$$f(e_0) = e'_0 = \sum_{i=0}^{n} A_i e_i, \quad f(e_1) = e'_1 = \sum_{i=1}^{n} B_i e_i, \quad f(e_i) = e'_i = [f(e_{i-1}), f(e_0)], \qquad 2 \le i \le n,$$

$A_0, A_i, B_j,\ (i, j = 1, ..., n)$ are complex numbers and $A_0 B_1 (A_0 + A_1 b) \neq 0$.

Proof.

Since a filiform Leibniz algebra is 2-generated (see Theorem 1.1.) it is sufficient to consider the adapted action of f on the generators e_0, e_1 :

$$f(e_0) = e'_0 = \sum_{i=0}^{n} A_i e_i, \quad f(e_1) = e'_1 = \sum_{i=0}^{n} B_i e_i.$$

Then $f(e_i) = [f(e_{i-1}), f(e_0)] = A_0^{i-2}(A_1 B_0 - A_0 B_1) e_i + \sum_{j=i+1}^{n} (*) e_j, \quad 2 \le i \le n.$

Note that $A_0 \neq 0,\ (A_1 B_0 - A_0 B_1) \neq 0$, otherwise $f(e_n) = 0$. The condition $A_0 B_1 (A_0 + A_1 b) \neq 0$ appears naturally since f is not singular.

Let now consider $[f(e_1), f(e_2)] = B_0\,(A_1\,B_0 - A_0\,B_1)\,e_3 + \sum_{j=4}^{n}(*)e_j$. Then for the basis $\{f(e_0), f(e_1), ..., f(e_n)\}$ to be adapted $B_0(A_1B_0 - A_0B_1) = 0$. But according to the observation above $(A_1B_0 - A_0B_1) \neq 0$. Therefore $B_0 = 0$. □

In G_{ad} we specify the following transformations called elementary:

$$\tau(a,b,c) = \begin{cases} \tau(e_0) = a\,e_0 + b\,e_1, & \\ \tau(e_1) = c\,e_1, & a\,c \neq 0, \\ \tau(e_{i+1}) = [\tau(e_i), \tau(e_0)], & 1 \leq i \leq n-1, \end{cases}$$

$$\sigma(a,k) = \begin{cases} \sigma(e_0) = e_0 + a\,e_k, & 2 \leq k \leq n, \\ \sigma(e_1) = e_1, & \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], & 1 \leq i \leq n-1, \end{cases}$$

$$\phi(c,k) = \begin{cases} \phi(e_0) = e_0, & \\ \phi(e_1) = e_1 + c\,e_k, & 2 \leq k \leq n, \\ \phi(e_{i+1}) = [\phi(e_i), \phi(e_0)], & 1 \leq i \leq n-1, \end{cases}$$

where $a, b, c \in \mathbb{C}$.

Proposition 1.2. *Let L be an algebra from $TLeib_{n+1}$, then any adapted transformation f can be represented as the composition:*

$$f = \phi(B_n, n) \circ \phi(B_{n-1}, n-1) \circ ... \circ \phi(B_2, 2) \circ \sigma(A_n, n) \circ \sigma(A_{n-1}, n-1) \circ ... \circ \sigma(A_2, 2) \circ \tau(A_0, A_1, B_1).$$

Proof. The proof is straightforward. □

Proposition 1.3. *The transformations*
$g = \phi(B_n, n) \circ \phi(B_{n-1}, n-1) \circ \phi(B_{n-2}, n-2) \circ \sigma(A_n, n) \circ \sigma(A_{n-1}, n-1) \circ \sigma(A_{n-2}, n-2) \circ \sigma(A_{n-3}, n-3)$, *if n even, and*
$g = \phi(B_n, n) \circ \phi(B_{n-1}, n-1) \circ \sigma(A_n, n) \circ \sigma(A_{n-1}, n-1) \circ \sigma(A_{n-2}, n-2)$, *for odd n does not change the structure constants of algebras from $TLeib_{n+1}$.*

Proof. Let us prove the assertion when n is even.

- Consider the transformation,

$$\sigma(A_{n-3}, n-3) = \begin{cases} \sigma(e_0) = e_0 + A_{n-3}\,e_{n-3}, & \\ \sigma(e_1) = e_1, & \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], & 1 \leq i \leq n-1. \end{cases}$$

  To show that the transformation $\sigma(A_{n-3}, n-3)$, does not change the structure constants, note that $\sigma\,(e_2) = e_2 + (*)\,A_{n-3}e_{n-1} + (**)A_{n-3}e_n$, $\sigma\,(e_3) = e_3 + (\star)\,A_{n-3}e_n$, and $\sigma\,(e_i) = e_i,\ \forall\ i \geq 4$, and by a simple computation one can see that the transformation $\sigma(A_{n-3}, n-3)$ does not change the structure constants.

- The transformation $\sigma(A_{n-2}, n-2)$, does not change the structure constants, because $\sigma(e_2) = e_2 - A_{n-2}\,(*)\,e_n$, and $\sigma(e_i) = e_i,\ \forall\ i \geq 3$.

- For the transformation $\sigma(A_{n-1}, n-1)$, it is enough show that $\sigma(e_2) = e_2$, note that $\sigma(e_2) = [\sigma(e_1), \sigma(e_0)] = e_2 + A_{n-1}[e_{n-1}, e_1]$, since n is even then $[e_{n-1}, e_1] = 0$, and hence $\sigma(e_2) = e_2$

It is easy to see that $\sigma(A_n, n)$, does not change the structure constants.

Analogously, we check that $\phi(B_k, k)$ does not change the structure constants, when $n-2 \leq k \leq n$. Consider the transformation

$$\phi(B_k, k) = \begin{cases} \phi(e_0) = e_0, & \\ \phi(e_1) = e_1 + B_k\, e_k, & n-2 \leq k \leq n, \\ \phi(e_{i+1}) = [\phi(e_i), \phi(e_0)], & 1 \leq i \leq n-1, \end{cases}$$

- If $k = n-2$, then

$$\phi(e_2) = e_2 + B_{n-2}e_{n-1},\ \phi(e_3) = e_3 + B_{n-2}e_n,\ \phi(e_i) = e_i,\ \text{where } i \geq 4.$$

A simple computation shows that $\phi(B_{n-2}, n-2)$ does not change the structure constants.
Note that $[\phi(e_1), \phi(e_2)] = [e_1 + B_{n-2}e_{n-2}, e_2 + B_{n-2}e_{n-1}]$

$$= [e_1, e_2] + B_{n-2}\,[e_1, e_{n-1}] + B_{n-2}\,[e_{n-2}, e_2] + B_{n-2}^2\,[e_{n-2}, e_{n-1}] = [e_1, e_2]\,,$$

(here $[e_1, e_{n-1}] = [e_{n-2}, e_2] = 0$, since n is even).

- If $k = n-1$, then we get $\phi(e_2) = e_2 + B_{n-1}e_n,\ \phi(e_i) = e_i$, where $i \geq 3$.

Consider the bracket

$$[\phi(e_0), \phi(e_1)] = -\phi(e_2) + b'_{0,1}\, e'_n,$$

and then

$$[e_0, e_1 + B_{n-1}e_{n-1}] = -e_2 - B_{n-1}e_n + b'_{0,1}\, e_n,$$

implies that

$$-e_2 + b_{0,1}\, e_n - B_{n-1}e_n = -e_2 - B_{n-1}e_n + b'_{0,1}\, e_n,$$

therefore $b'_{0,1} = b_{0,1}$.
The chain of equalities

$$\begin{aligned} [\phi(e_1), \phi(e_1)] &= b'_{1,1}\, e'_n, \\ [e_1 + B_{n-1}e_{n-1}, e_1 + B_{n-1}e_{n-1}] &= b'_{1,1}\, e_n, \\ [e_1, e_1] + B_{n-1}[e_1, e_{n-1}] + B_{n-1}[e_{n-1}, e_1] &= b'_{1,1}\, e_n,\ \text{show that} \quad b'_{1,1} = b_{1,1}. \end{aligned}$$

One easily can see that

$$[\phi(e_1), \phi(e_2)] = [e_1 + B_{n-1}e_{n-1}, e_2 + B_{n-1}e_n] = [e_1, e_2] + B_{n-1}[e_1, e_n] + B_{n-1}[e_{n-1}, e_2] = [e_1, e_2].$$

- If $k = n$, it is obvious.

□

The following lemma from [10] keeps track the behavior some of the structure constants under the adapted base change.

Lemma 1.1. *Let $\{e_0, e_1, ..., e_n\} \longrightarrow \{e'_0, e'_1, ..., e'_n\}$ be an adapted base change, $b_{0,0}, b_{0,1}, b_{1,1}, ...$ and $b'_{0,0}, b'_{0,1}, b'_{1,1}, ...$ be the respective structure constants. Then for $b'_{0,0}, b'_{0,1}$ and $b'_{1,1}$ one has*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-2} B_1 (A_0 + A_1\, b)}, \quad b'_{0,1} = \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^{n-2}(A_0 + A_1\, b)}, \quad b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^{n-2}(A_0 + A_1\, b)}.$$

The next sections deal with the classification problem of $TLeib_n$ in dimensions 7 and 8. Here, to classify algebras from $TLeib_7$ and $TLeib_8$ we represent them as a disjoint union of their subsets. Some of these subsets turn out to be single orbits, and the others contain infinitely many orbits. In the last case, we give invariant functions to discern the orbits.

2 The description of $TLeib_n$, $n = 7, 8$.

2.1 Isomorphism criterion for $TLeib_7$

Any algebra L from $TLeib_7$ can be represented as one dimensional central extension ($C(L) =< e_6 >$) of 6-dimensional filiform Lie algebra with adapted basis $\{e_0, e_1, ..., e_5\}$ (see Theorem 1.1) and on the adapted basis $\{e_0, e_1, ..., e_6\}$ the class $TLeib_7$ can be represented as follows:

$$TLeib_7 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq 5, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq 5, \\ [e_0, e_0] = b_{0,0}e_6, \\ [e_0, e_1] = -e_2 + b_{0,1}e_6, \\ [e_1, e_1] = b_{1,1}e_6, \\ [e_1, e_2] = -[e_2, e_1] = a_{1,4}\, e_4 + a_{1,5}\, e_5 + b_{1,2}e_6, \\ [e_1, e_3] = -[e_3, e_1] = a_{1,4}\, e_5 + b_{1,3}e_6, \\ [e_1, e_4] = -[e_4, e_1] = -a_{2,5}\, e_5 + b_{1,4}e_6, \\ [e_2, e_3] = -[e_3, e_2] = a_{2,5}\, e_5 + b_{2,3}e_6, \\ [e_1, e_5] = -[e_5, e_1] = b_{1,5}e_6, \\ [e_2, e_4] = -[e_4, e_2] = b_{2,4}e_6. \end{cases}$$

The next lemma specifies the set of structure constants of algebras from $TLeib_7$.

Lemma 2.1. *The structure constants of algebras from $TLeib_7$ satisfy the following constraints:*

$$1.\ b_{1,3} = a_{1,5}, \quad 2.\ b_{1,4} = a_{1,4} - b_{2,3}, \quad \textit{and} \quad 3.\ b_{1,5} = b_{2,4} = a_{2,5} = 0.$$

Proof. The relations easily can be found by applying the Leibniz identity to the triples of the basis vectors $\{e_0, e_1, e_2\}$, $\{e_0, e_1, e_3\}$ $\{e_0, e_2, e_3\}$, $\{e_0, e_1, e_4\}$, and $\{e_1, e_2, e_3\}$. □

Further unifying the above table of multiplication we rewrite it via new parameters $c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{2,3}$ as follows:

$$TLeib_7 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq 5, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq 5, \\ [e_0, e_0] = c_{0,0}e_6, \\ [e_0, e_1] = -e_2 + c_{0,1}e_6, \\ [e_1, e_1] = c_{1,1}e_6, \\ [e_1, e_2] = -[e_2, e_1] = c_{1,2}\, e_4 + c_{1,3}\, e_5 + c_{1,4}e_6, \\ [e_1, e_3] = -[e_3, e_1] = c_{1,2}\, e_5 + c_{1,3}e_6, \\ [e_1, e_4] = -[e_4, e_1] = (c_{1,2} - c_{2,3})\, e_6, \\ [e_2, e_3] = -[e_3, e_2] = c_{2,3}\, e_6. \end{cases}$$

An algebra from $TLeib_7$ with parameters $c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{2,3}$ is denoted by $L(C)$, where $C = (c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{2,3})$.

The next theorem represents the action of the adapted base change to the parameters $c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{2,3}$ of an algebra from $TLeib_7$.

Theorem 2.1. (*Isomorphism criterion for $TLeib_7$*)

Two filiform Leibniz algebras $L(C)$ and $L(C')$, where $C = (c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{2,3})$ and $C' = (c'_{0,0}, c'_{0,1}, c'_{1,1}, c'_{1,2}, c'_{1,3}, c'_{1,4}, c'_{2,3})$, from $TLeib_7$ are isomorphic if and only if there exist $A_0, A_1, B_1, B_2, B_3 \in \mathbb{C}$: such that $A_0 B_1 \neq 0$ and the following equalities hold:

$$
\begin{aligned}
c'_{0,0} &= \frac{{A_0}^2 c_{0,0} + A_0 A_1 c_{0,1} + {A_1}^2 c_{1,1}}{{A_0}^5 B_1}, && (1)\\
c'_{0,1} &= \frac{2\,A_1 c_{1,1} + A_0 c_{0,1}}{{A_0}^5}, && (2)\\
c'_{1,1} &= \frac{B_1 c_{1,1}}{{A_0}^5}, && (3)\\
c'_{1,2} &= \frac{B_1 c_{1,1}}{{A_0}^5}, && (4)\\
c'_{1,3} &= \frac{B_1\left(A_0 c_{1,3} + 2\,A_1\,{c_{1,2}}^2\right)}{{A_0}^4}, && (5)\\
c'_{1,4} &= \frac{{A_0}^2\,{B_1}^2 c_{1,4} + {A_0}^2({B_2}^2 - 2B_1\,B_3)\,c_{2,3} + A_1\,{B_1}^2(5\,c_{1,2} - c_{2,3})\,(A_0\,c_{1,3} - B_1\,{c_{1,2}}^2)}{{A_0}^6 B_1}, && (6)\\
c'_{2,3} &= \frac{B_1 c_{2,3}}{{A_0}^2}. && (7)
\end{aligned}
$$

Proof. "If" part. The equations (1)–(3) occur due to Lemma 1.1 (remind that in this case n is even therefore $b = 0$).

Notice that according to the Proposition 1.2 and 1.3 the adapted transformation in $TLeib_7$ can be taken in the form

$$
\begin{cases}
e'_0 = f(e_0) = A_0\,e_0 + A_1\,e_1, & \\
e'_1 = f(e_1) = B_1\,e_1 + B_2\,e_2 + B_3\,e_3, & \\
e'_{i+1} = f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \le i \le n-1,
\end{cases}
$$

where $A_0\,B_1 \neq 0$ or more precisely as follows:

$$
\begin{aligned}
e'_0 &= A_0 e_0 + A_1 e_1,\\
e'_1 &= B_1\,e_1 + B_2 e_2 + B_3 e_3,\\
e'_2 &= A_0 B_1\,e_2 + A_0 B_2 e_3 + (A_0 B_3 - A_1 B_2 c_{1,2})\,e_4 - A_1\,(B_2 c_{1,3} + B_3 c_{1,2})\,e_5\\
&\quad + A_1\,(B_1 c_{1,1} - B_2 c_{1,4} - B_3 c_{1,3})\,e_6,\\
e'_3 &= A_0^2 B_1\,e_3 + \left(A_0^2 B_2 - A_0 A_1 B_1 c_{1,2}\right)e_4 + (A_0^2 B_3 - 2\,A_0 A_1 B_2 c_{1,2} - A_0 A_1 B_1 c_{1,3})e_5\\
&\quad + (-A_1\left(-A_1 B_2 c_{1,2}^2 + A_1 B_2 c_{1,2} c_{2,3} + A_0 B_1 c_{1,4} + 2\,A_0 B_2 c_{1,3} + 2\,A_0 B_3 c_{1,2} - A_0 B_3 c_{2,3}\right))e_6, && (8)\\
e'_4 &= A_0^3 B_1\,e_4 + (A_0^3 B_2 - 2\,A_0^2 A_1 B_1 c_{1,2})e_5 +\\
&\quad (A_0^3 B_3 - 2\,A_1 A_0^2 B_1 c_{1,3} + A_0 A_1^2 B_1 b_{1,2}^2 - A_0 {A_1}^2 B_1 c_{1,2} c_{2,3} - 3\,A_1 {A_0}^2 B_2 c_{1,2} + A_0^2 A_1 B_2 c_{2,3})e_6,\\
e'_5 &= A_0^4\,B_1\,e_5 + (A_0^4 B_2 - 3\,A_1 A_0^3 B_1 c_{1,2} + A_1 A_0^3 B_1 b_{2,3})e_6,\\
e'_6 &= A_0^5\,B_1\,e_6.
\end{aligned}
$$

Consider

$$
\begin{aligned}
[e'_1, e'_2] \;=\;& A_0\,{B_1}^2\,(c_{1,2}e_4 + c_{1,3}e_5 + c_{1,4}e_6) + A_0 B_1 B_2\,(c_{1,2}e_5 + c_{1,3}e_6) + B_1(A_2 B_1 c_{1,2} - A_1 B_2 c_{1,2} +\\
& A_0 B_3)\,(c_{1,2} - c_{2,3})\,e_6 + A_0\,{B_2}^2 c_{2,3} e_6 - A_0 B_1 B_3 c_{2,3} e_6 = c'_{1,2}e'_4 + c'_{1,3}e'_5 + c'_{1,4}e'_6.
\end{aligned}
$$

Now bearing in mind (8) and equating the coefficients of e_4, e_5 and e_6 we get the equalities (4), (5) and (6), respectively. The last equality follows from

$$
[f(e_2), f(e_3)] = c'_{2,3} f(e_6) \implies {A_0}^3 {B_1}^2 c_{2,3} e_6 = c'_{2,3} {A_0}^5 B_1 e_6 \implies c'_{2,3} = \frac{B_1\,c_{2,3}}{A_0^2}.
$$

"Only if" part. Let the equalities (1)–(7) hold. Then the above base change is adapted and it transforms $L(C)$ to $L(C')$.

Indeed,

$$\begin{aligned}
[e'_0, e'_0] &= [A_0e_0 + A_1e_1,\ A_0e_0 + A_1e_1] \\
&= A_0^2[e_0, e_0] + A_0A_1[e_0, e_1] + A_0A_1[e_1, e_0] + A_1^2[e_1, e_1] \\
&= \left(A_0^2c_{0,0} + A_0A_1c_{0,1} + A_1^2c_{1,1}\right) e_6 = c'_{0,0}A_0^5B_1e_6 = c'_{0,0}e'_6.
\end{aligned}$$

$$\begin{aligned}
[e'_0, e'_1] &= [A_0e_0 + A_1e_1,\ B_1e_1 + B_2e_2 + B_3e_3] \\
&= A_0B_1\,(-e_2 + c_{0,1}e_6) - A_0B_2e_3 - A_0B_3e_4 + A_1B_1c_{1,1}e_6 + A_1B_2(c_{1,2}e_4 + c_{1,3}e_5 + c_{1,4}e_6) + \\
&\quad A_1B_3(c_{1,2}e_5 + c_{1,3}e_6) \\
&= -(A_0B_1\,e_2 + A_0B_2e_3 + (A_0B_3 - A_1B_2c_{1,2})\,e_4 - A_1\,(B_2c_{1,3} + B_3c_{1,2})\,e_5 + \\
&\quad A_1\,(B_1c_{1,1} - B_2c_{1,4} - B_3c_{1,3})\,e_6) + B_1\,(A_0c_{0,1} + 2\,A_1c_{1,1})\,e_6 \\
&= -e'_2 + A_0^5B_1c'_{0,1}e_5 = -e'_2 + c'_{0,1}e'_5.
\end{aligned}$$

$$[e'_1, e'_1] = [B_1e_1 + B_2e_2 + B_3e_3, B_1e_1 + B_2e_2 + B_3e_3] = [B_1e_1, B_1e_1] = B_1^2c_{1,1}e_6 = A_0^5B_1c'_{1,1}e_6 = c'_{1,1}e'_6.$$

The brackets $[e'_1, e'_2], [e'_1, e'_3], [e'_1, e'_4]$ and $[e'_2, e'_3]$ can be gotten similarly. □

The next section deals with the classification problem of $TLeib_7$.

2.1.1 Isomorphism classes in $TLeib_7$

In this subsection we give a list of all algebras from $TLeib_7$.

Represent $TLeib_7$ as a union of the following subsets:

$U_7^1 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} \neq 0\};$
$U_7^2 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} = 0,\ c_{0,1} \neq 0\};$
$U_7^3 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = 0,\ c_{1,2} \neq 0,\ c_{0,0} \neq 0\};$
$U_7^4 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = 0,\ c_{1,2} \neq 0,\ c_{0,0} = 0\};$
$U_7^5 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = c_{1,2} = 0,\ c_{1,3} \neq 0\};$
$U_7^6 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = c_{1,2} = c_{1,3} = 0,\ c_{0,0} \neq 0\};$
$U_7^7 = \{L(C) \in TLeib_7\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = c_{1,2} = c_{1,3} = c_{0,0} = 0\};$
$U_7^8 = \{L(C) \in TLeib_7\ : c_{2,3} = 0,\ c_{1,2} \neq 0,\ c_{1,1} \neq 0\};$
$U_7^9 = \{L(C) \in TLeib_7\ : c_{2,3} = 0,\ c_{1,2} \neq 0,\ c_{1,1} = 0,\ c_{0,1} \neq 0\};$
$U_7^{10} = \{L(C) \in TLeib_7\ : c_{2,3} = 0,\ c_{1,2} \neq 0,\ c_{1,1} = c_{0,1} = 0,\ c_{0,0} \neq 0\};$
$U_7^{11} = \{L(C) \in TLeib_7\ : c_{2,3} = 0,\ c_{1,2} \neq 0,\ c_{1,1} = c_{0,1} = c_{0,0} = 0,\ 4\,c_{1,4}c_{1,2} - 5\,c_{1,3}^2 \neq 0\};$
$U_7^{12} = \{L(C) \in TLeib_7\ : c_{2,3} = 0,\ c_{1,2} \neq 0,\ c_{1,1} = c_{0,1} = c_{0,0} = 4\,c_{1,4}c_{1,2} - 5\,c_{1,3}^2 = 0\};$
$U_7^{13} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = 0,\ c_{1,1} \neq 0,\ c_{1,4} \neq 0\};$
$U_7^{14} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = 0,\ c_{1,1} \neq 0,\ c_{1,4} = 0,\ c_{1,3} \neq 0\};$
$U_7^{15} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = 0,\ c_{1,1} \neq 0,\ c_{1,4} = c_{1,3} = 0,\ 4\ c_{0,0}c_{1,1} - c_{0,1}^2 \neq 0\};$
$U_7^{16} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = 0,\ c_{1,1} \neq 0,\ c_{1,4} = c_{1,3} = 4\,c_{0,0}c_{1,1} - c_{0,1}^2 = 0\};$
$U_7^{17} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = 0,\ c_{0,1} \neq 0,\ c_{1,4} \neq 0\};$
$U_7^{18} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = 0,\ c_{0,1} \neq 0,\ c_{1,4} = 0,\ c_{1,3} \neq 0\};$
$U_7^{19} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = 0,\ c_{0,1} \neq 0,\ c_{1,4} = c_{1,3} = 0\};$
$U_7^{20} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = 0,\ c_{0,0} \neq 0,\ c_{1,4} \neq 0\};$
$U_7^{21} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = 0,\ c_{0,0} \neq 0,\ c_{1,4} = 0,\ c_{1,3} \neq 0\};$
$U_7^{22} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = 0,\ c_{0,0} \neq 0,\ c_{1,4} = c_{1,3} = 0\};$
$U_7^{23} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = c_{0,0} = 0,\ c_{1,4} \neq 0,\ c_{1,3} \neq 0\};$
$U_7^{24} = \{L(C) \in TLeib_7\ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = c_{0,0} = 0,\ c_{1,4} \neq 0,\ c_{1,3} = 0\};$

$U_7^{25} = \{L(C) \in TLeib_7 \ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = c_{0,0} = c_{1,4} = 0, \ c_{1,3} \neq 0\};$
$U_7^{26} = \{L(C) \in TLeib_7 \ : c_{2,3} = c_{1,2} = c_{1,1} = c_{0,1} = c_{0,0} = c_{1,4} = c_{1,3} = 0\}.$

Proposition 2.1.

1. *Two algebras $L(C)$ and $L(C')$ from U_7^1 are isomorphic, if and only if*

$$\left(\frac{c'_{2,3}}{c'_{1,1}}\right)^8 (4\,c'_{0,0}c'_{1,1} - {c'}_{0,1}^2) = \left(\frac{c_{2,3}}{c_{1,1}}\right)^8 (4\,c_{0,0}c_{1,1} - c_{0,1}^2),$$

$$\frac{c'_{1,2}}{c'_{2,3}} = \frac{c_{1,2}}{c_{2,3}} \quad \text{and} \quad \frac{\left(c'_{1,3}c'_{1,1} - c'_{0,1}{c'}_{1,2}^2\right)^3}{{c'}_{2,3}^2{c'}_{1,1}^4} = \frac{\left(c_{1,3}c_{1,1} - c_{0,1}c_{1,2}^2\right)^3}{c_{2,3}^2c_{1,1}^4}.$$

2. *For any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, there exists $L(C) \in U_7^1$:*

$$\left(\frac{c_{2,3}}{c_{1,1}}\right)^8 (4\,c_{0,0}c_{1,1} - c_{0,1}^2) = \lambda_1, \quad \frac{c_{1,2}}{c_{2,3}} = \lambda_2, \quad \frac{\left(c_{1,3}c_{1,1} - c_{0,1}c_{1,2}^2\right)^3}{c_{2,3}^2c_{1,1}^4} = \lambda_3.$$

Then orbits in U_7^1 can be parameterized as $L\left(\lambda_1, 0, 1, \lambda_2, \lambda_3, 0, 1\right), \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

Proof. 1. "If" part due to Theorem 2.1 if one substitutes the expressions for $c'_{0,0}$, $c'_{0,1}$, $c'_{1,1}$, $c'_{1,2}$, $c'_{1,3}$, $c'_{1,4}$, $c'_{2,3}$:

$$\left(\frac{c'_{2,3}}{c'_{1,1}}\right)^8 (4\,c'_{0,0}c'_{1,1} - {c'}_{0,1}^2) = \left(\frac{c_{2,3}}{c_{1,1}}\right)^8 (4\,c_{0,0}c_{1,1} - c_{0,1}^2),$$

$$\frac{c'_{1,2}}{c'_{2,3}} = \frac{c_{1,2}}{c_{2,3}}, \quad \frac{\left(c'_{1,3}c'_{1,1} - c'_{0,1}{c'}_{1,2}^2\right)^3}{{c'}_{2,3}^2{c'}_{1,1}^4} = \frac{\left(c_{1,3}c_{1,1} - c_{0,1}c_{1,2}^2\right)^3}{c_{2,3}^2c_{1,1}^4}.$$

" Only if " part. Let the equalities

$$\left(\frac{c'_{2,3}}{c'_{1,1}}\right)^8 (4\,c'_{0,0}c'_{1,1} - {c'}_{0,1}^2) = \left(\frac{c_{2,3}}{c_{1,1}}\right)^8 (4\,c_{0,0}c_{1,1} - c_{0,1}^2),$$

$$\frac{c'_{1,2}}{c'_{2,3}} = \frac{c_{1,2}}{c_{2,3}}, \quad \frac{\left(c'_{1,3}c'_{1,1} - c'_{0,1}{c'}_{1,2}^2\right)^3}{{c'}_{2,3}^2{c'}_{1,1}^4} = \frac{\left(c_{1,3}c_{1,1} - c_{0,1}c_{1,2}^2\right)^3}{c_{2,3}^2c_{1,1}^4}$$

hold.

Consider the base change (8) in Theorem 2.1 with

$$A_1 = \frac{-A_0c_{0,1}}{2\,c_{1,1}}, \ B_1 = \frac{{A_0}^2}{c_{2,3}},$$

and

$$B_3 = \frac{1}{8\,{A_0}^2{c_{1,1}}^2{c_{2,3}}^2}(4\,{c_{1,1}}^2\left({A_0}^4c_{1,4} + 4\,{B_2}^2{c_{2,3}}^3\right) + {A_0}^4\,(5\,c_{1,2} - c_{2,3})\left({c_{0,1}}^2{c_{1,2}}^2 - 2\,c_{0,1}c_{1,1}c_{1,3}\right)).$$

This changing leads $L(C)$ into

$$L\left(\left(\frac{c_{2,3}}{c_{1,1}}\right)^8 (4\,c_{0,0}c_{1,1} - c_{0,1}^2), 0, 1, \frac{c_{1,2}}{c_{2,3}}, \frac{\left(c_{1,3}c_{1,1} - c_{0,1}c_{1,2}^2\right)^3}{c_{2,3}^2c_{1,1}^4}, 0, 1\right).$$

An analogous base change with "dash" transforms $L(C')$ into

$$L\left(\left(\frac{c'_{2,3}}{c'_{1,1}}\right)^8 (4\,c'_{0,0}c'_{1,1}-c'^2_{0,1}),0,1,\frac{c'_{1,2}}{c'_{2,3}},\frac{\left(c'_{1,3}c'_{1,1}-c'_{0,1}c'^2_{1,2}\right)^3}{c'^2_{2,3}c'^4_{1,1}},0,1\right).$$

Since

$$\left(\frac{c'_{2,3}}{c'_{1,1}}\right)^8 (4\,c'_{0,0}c'_{1,1}-c'^2_{0,1})=\left(\frac{c_{2,3}}{c_{1,1}}\right)^8 (4\,c_{0,0}c_{1,1}-c^2_{0,1}),\quad \frac{c'_{1,2}}{c'_{2,3}}=\frac{c_{1,2}}{c_{2,3}}$$

and

$$\frac{\left(c'_{1,3}c'_{1,1}-c'_{0,1}c'^2_{1,2}\right)^3}{c'^2_{2,3}c'^4_{1,1}}=\frac{\left(c_{1,3}c_{1,1}-c_{0,1}c^2_{1,2}\right)^3}{c^2_{2,3}c^4_{1,1}},$$

the algebras are isomorphic.

2. Obvious.

□

Proposition 2.2.

1. *Two algebras $L(C)$ and $L(C')$ from U_7^2 are isomorphic, if and only if*

$$\frac{c'_{1,2}}{c'_{2,3}}=\frac{c_{1,2}}{c_{2,3}},\quad \frac{\left(c'_{1,3}c'_{0,1}-2\,c'_{0,0}c'^2_{1,2}\right)^4}{c'^4_{2,3}c'^5_{0,1}}=\frac{\left(c_{1,3}c_{0,1}-2\,c_{0,0}c^2_{1,2}\right)^4}{c^4_{2,3}c^5_{0,1}}.$$

2. *For any $\lambda_1,\lambda_2\in\mathbb{C}$, there exists $L(C)\in U_7^2$:*

$$\frac{c_{1,2}}{c_{2,3}}=\lambda_1,\ \frac{\left(c_{1,3}c_{0,1}-2\,c_{0,0}c^2_{1,2}\right)^4}{c^4_{2,3}c^5_{0,1}}=\lambda_2.$$

The orbits in U_7^2 can be parameterized as $L\left(0,1,0,\lambda_1,\lambda_2,0,1\right),\ \lambda_1,\lambda_2\in\mathbb{C}$.

Proof. The proof is similar that of Proposition 2.1, where we put

$$A_1=-\frac{A_0c_{0,0}}{c_{0,1}},\ B_1=\frac{{A_0}^2}{c_{2,3}},$$

and

$$B_3=\frac{({c_{0,1}}^2\left({A_0}^4c_{1,4}+{B_2}^2{c_{2,3}}^3\right)+{A_0}^4(c_{0,0}\left(c_{0,0}{c_{1,2}}^2-c_{1,3}c_{0,1}\right)(5\,c_{1,2}-c_{2,3})))}{2\,{A_0}^2{c_{0,1}}^2{c_{2,3}}^2}.$$

□

Since the proving of the next coming Propositions 2.3 – 2.13 are similar those of Propositions 2.1 and 2.2 we decided to omit the details of them. "If" parts of them follow from Theorem 1.2, for "Only if" part we just give the respective values of the coefficients A_0,A_1,B_1,B_2 and B_3 in the base change (8). Note that if no value is given then it is considered as an arbitrary.

Proposition 2.3.

1. *Two algebras $L(C)$ and $L(C')$ from U_7^3 are isomorphic, if and only if* $\dfrac{c'_{1,2}}{c'_{2,3}}=\dfrac{c_{1,2}}{c_{2,3}}$.

2. *For any $\lambda\in\mathbb{C}^*$, there exists $L(C)$ from U_7^3 :* $\dfrac{c_{1,2}}{c_{2,3}}=\lambda$.

Then orbits in U_7^3 *can be parameterized as* $L\,(1,0,0,\lambda,0,0,1)\,,\ \ \lambda\in\mathbb{C}^*$.

Proof. The respective values of coefficients are:

$$A_1=\frac{-A_0c_{1,3}}{2\,{c_{1,2}}^2},\ \ B_1=\frac{{A_0}^2}{c_{2,3}}$$

and

$$B_3=\frac{{A_0}^4\,\left(4\,c_{1,4}{c_{1,2}}^2-5\,{c_{1,3}}^2c_{1,2}+{c_{1,3}}^2c_{2,3}\right)+4\,{B_2}^2{c_{2,3}}^3{c_{1,2}}^2}{8\,{A_0}^2{c_{1,2}}^2{c_{2,3}}^2}.$$

□

Proposition 2.4.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^4 *are isomorphic, if and only if* $\dfrac{c'_{1,2}}{c'_{2,3}}=\dfrac{c_{1,2}}{c_{2,3}}$.

2. *For any* $\lambda\in\mathbb{C}^*$, *there exists* $L(C)$ *from* U_7^4 : $\dfrac{c_{1,2}}{c_{2,3}}=\lambda$.

   Then orbits in U_7^4 *can be parameterized as* $L\,(0,0,0,\lambda,0,0,1)\,,\ \ \lambda\in\mathbb{C}^*$.

Proof. Here

$$A_1=\frac{-A_0c_{1,3}}{2\,{c_{1,2}}^2},\ \ B_1=\frac{{A_0}^2}{c_{2,3}}\ \ \text{and}\ \ B_3=\frac{{A_0}^4\,\left(4\,c_{1,4}{c_{1,2}}^2-5\,{c_{1,3}}^2c_{1,2}+{c_{1,3}}^2c_{2,3}\right)+4\,{B_2}^2{c_{2,3}}^3{c_{1,2}}^2}{8\,{A_0}^2{c_{1,2}}^2{c_{2,3}}^2}.$$

□

Proposition 2.5.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^5 *are isomorphic, if and only if*

$$\frac{{c'_{2,3}}^6c'_{0,0}}{{c'_{1,3}}^5}=\frac{{c_{2,3}}^6c_{0,0}}{{c_{1,3}}^5}.$$

2. *For any* $\lambda\in\mathbb{C}$, *there exists* $L(C)\in U_7^5$: $\dfrac{{c_{2,3}}^6c_{0,0}}{{c_{1,3}}^5}=\lambda$.

   Then orbits in U_7^5 *can be parameterized as* $L\,(\lambda,0,0,0,1,0,1)\,,\lambda\in\mathbb{C}$.

Proof. For this case:

$$A_0=\frac{c_{1,3}}{c_{2,3}},\ \ B_1=\frac{{c_{1,3}}^2}{{c_{2,3}}^3}\ \ \text{and}\ \ B_3=\frac{{c_{1,3}}^4c_{1,4}+{B_2}^2{c_{2,3}}^7-A_1{c_{1,3}}^4{c_{2,3}}^2}{2\,{c_{1,3}}^2{c_{2,3}}^4}.$$

□

Proposition 2.6.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^8 *are isomorphic, if and only if*

$$\frac{\left(4\,c'_{0,0}{c'_{1,2}}^4-2\,c'_{1,3}c'_{0,1}{c'_{1,2}}^2+{c'_{1,3}}^2c'_{1,1}\right)^3}{{c'_{1,2}}^4{c'_{1,1}}^5}=\frac{\left(4\,c_{0,0}{c_{1,2}}^4-2\,c_{1,3}c_{0,1}{c_{1,2}}^2+{c_{1,3}}^2c_{1,1}\right)^3}{{c_{1,2}}^4{c_{1,1}}^5},$$

$$\frac{\left(c'_{0,1}{c'_{1,2}}^2-c_{1,3}c_{1,1}\right)^3}{{c'_{1,2}}^2{c'_{1,1}}^4}=\frac{\left(c_{0,1}{c_{1,2}}^2-c_{1,3}c_{1,1}\right)^3}{{c_{1,2}}^2{c_{1,1}}^4}\ \ \textit{and}\ \ \frac{\left(4\,c'_{1,4}c'_{1,2}-5\,{c'_{1,3}}^2\right)^3}{{c'_{1,2}}^4{c'_{1,1}}^2}=\frac{\left(4\,c_{1,4}c_{1,2}-5\,{c_{1,3}}^2\right)^3}{{c_{1,2}}^4{c_{1,1}}^2}.$$

2. *For any* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, *there exists* $L(C) \in U_7^8$:
$$\frac{\left(4\,c_{0,0}{c_{1,2}}^4 - 2\,c_{1,3}c_{0,1}{c_{1,2}}^2 + {c_{1,3}}^2c_{1,1}\right)^3}{{c_{1,2}}^4{c_{1,1}}^5} = \lambda_1,\ \frac{\left(c_{0,1}{c_{1,2}}^2 - c_{1,3}c_{1,1}\right)^3}{{c_{1,2}}^2{c_{1,1}}^4} = \lambda_2,\ \frac{\left(4\,c_{1,4}c_{1,2} - 5\,{c_{1,3}}^2\right)^3}{{c_{1,2}}^4{c_{1,1}}^2} = \lambda_3.$$

Then orbits in U_7^8 *can be parameterized as* $L\left(\lambda_1, \lambda_2, 1, 1, 0, \lambda_3, 0\right),\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

Proof. We put here,
$$A_1 = \frac{-A_0c_{1,3}}{2\,{c_{1,2}}^2},\ B_1 = \frac{{A_0}^2}{c_{1,2}}.$$
□

Proposition 2.7.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^9 *are isomorphic, if and only if*
$$\frac{\left(2\,c'_{0,0}{c'_{1,2}}^2 - c'_{1,3}c'_{0,1}\right)^4}{{c'_{1,2}}^4{c'_{0,1}}^5} = \frac{\left(2\,c_{0,0}{b_{1,2}}^2 - c_{1,3}c_{0,1}\right)^4}{{c_{1,2}}^4{c_{0,1}}^5}\ \ \textit{and}\ \ \frac{\left(4\,c'_{1,4}c'_{1,2} - 5\,{c'_{1,3}}^2\right)^2}{{c'_{1,2}}^4c'_{0,1}} = \frac{\left(4\,c_{1,4}c_{1,2} - 5\,{c_{1,3}}^2\right)^2}{{c_{1,2}}^4c_{0,1}}.$$

2. *For any* $\lambda_1, \lambda_2 \in \mathbb{C}$, *there exists* $L(C) \in U_7^9$:
$$\frac{\left(2\,c_{0,0}{b_{1,2}}^2 - c_{1,3}c_{0,1}\right)^4}{{c_{1,2}}^4{c_{0,1}}^5} = \lambda_1\ \ \textit{and}\ \ \frac{\left(4\,c_{1,4}c_{1,2} - 5\,{c_{1,3}}^2\right)^2}{{c_{1,2}}^4c_{0,1}} = \lambda_2.$$

Then orbits in U_7^9 *can be parameterized as* $L\left(\lambda_1, 1, 0, 1, 0, \lambda_2, 0\right),\ \lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. For this case we put
$$A_1 = \frac{-A_0c_{1,3}}{2\,{c_{1,2}}^2},\ \text{and}\ B_1 = \frac{{A_0}^2}{c_{1,2}}.$$
□

Proposition 2.8.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^{10} *are isomorphic, if and only if*
$$\frac{\left(4\,c'_{1,4}c'_{1,2} - 5\,{c'_{1,3}}^2\right)^5}{{c'_{1,2}}^{12}{c'_{0,0}}^2} = \frac{\left(4\,c_{1,4}c_{1,2} - 5\,{c_{1,3}}^2\right)^5}{{c_{1,2}}^{12}{c_{0,0}}^2}.$$

2. *For any* $\lambda \in \mathbb{C}$, *there exists* $L(C) \in U_7^{10}$: $\dfrac{\left(4\,c_{1,4}c_{1,2} - 5\,{c_{1,3}}^2\right)^5}{{c_{1,2}}^{12}{c_{0,0}}^2} = \lambda.$

Then orbits in U_7^{10} *can be parameterized as* $L\left(1, 0, 0, 1, 0, \lambda, 0\right),\ \lambda \in \mathbb{C}$.

Proof. Here,
$$A_1 = \frac{-A_0c_{1,3}}{2\,{c_{1,2}}^2}\ \ \text{and}\ \ B_1 = \frac{{A_0}^2}{c_{1,2}}.$$
□

Proposition 2.9.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^{13} *are isomorphic, if and only if*
$$\left(\frac{c'_{1,4}}{c'_{1,1}}\right)^8\left(4\,c'_{0,0}c'_{1,1} - {c'_{0,1}}^2\right) = \left(\frac{c_{1,4}}{c_{1,1}}\right)^8\left(4\,c_{0,0}c_{1,1} - {c_{0,1}}^2\right),\quad \frac{c'_{1,1}c'_{1,3}}{{c'_{1,4}}^2} = \frac{c_{1,1}c_{1,3}}{{c_{1,4}}^2}.$$

2. *For any* $\lambda_1, \lambda_2 \in \mathbb{C}$, *there exists* $L(C) \in U_7^{13}:\quad \left(\frac{c_{1,4}}{c_{1,1}}\right)^8 \left(4\,c_{0,0}c_{1,1} - {c_{0,1}}^2\right) = \lambda_1, \quad \frac{c_{1,1}c_{1,3}}{{c_{1,4}}^2} = \lambda_2.$

   Then orbits in U_7^{13} *can be parameterized as* $L\left(\lambda_1, 0, 1, 0, \lambda_2, 1, 0\right),\ \lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. We put

$$A_0 = \frac{c_{1,1}}{c_{1,4}},\ A_1 = \frac{-c_{0,1}}{2\,c_{1,4}},\ \text{and } B_1 = \frac{{c_{1,1}}^4}{{c_{1,4}}^5}.$$

□

Proposition 2.10.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^{14} *are isomorphic, if and only if*

$$\left(\frac{{c'}_{1,3}}{{c'}_{1,1}}\right)^4 \left(4\,c'_{0,0}c'_{1,1} - {{c'}_{0,1}}^2\right) = \left(\frac{c_{1,3}}{c_{1,1}}\right)^4 \left(4\,c_{0,0}c_{1,1} - {c_{0,1}}^2\right).$$

2. *For any* $\lambda \in \mathbb{C}$, *there exists* $L(C) \in U_7^{14}:\quad \left(\frac{c_{1,3}}{c_{1,1}}\right)^4 \left(4\,c_{0,0}c_{1,1} - {c_{0,1}}^2\right) = \lambda.$

   Then orbits in U_7^{14} *can be parameterized as* $L\left(\lambda, 0, 1, 0, 1, 0, 0\right),\ \lambda \in \mathbb{C}$.

Proof. Here, we take

$$A_1 = \frac{-A_0 c_{0,1}}{2\,c_{1,1}},\ \text{and } B_1 = \frac{{A_0}^5}{c_{1,1}}.$$

□

Proposition 2.11.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^{17} *are isomorphic, if and only if* $\left(\frac{c'_{1,3}}{c'_{1,4}}\right)^4 c'_{0,1} = \left(\frac{c_{1,3}}{c_{1,4}}\right)^4 c_{0,1}.$

2. *For any* $\lambda \in \mathbb{C}$, *there exists* $L(C) \in U_7^{17}: \left(\frac{c_{1,3}}{c_{1,4}}\right)^4 c_{0,1} = \lambda.$

   Then orbits in U_7^{17} *can be parameterized as* $L\left(0, 1, 0, 0, \lambda, 1, 0\right),\ \lambda \in \mathbb{C}$.

Proof. Take

$$A_1 = -\frac{A_0 c_{0,0}}{c_{0,1}},\ \text{and } B_1 = \frac{{A_0}^4}{c_{1,4}}.$$

□

Proposition 2.12.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_7^{20} *are isomorphic, if and only if* $\frac{c'_{0,0}{{c'}_{1,3}}^7}{{{c'}_{1,4}}^6} = \frac{c_{0,0}{c_{1,3}}^7}{{c_{1,4}}^6}.$

2. *For any* $\lambda \in \mathbb{C}$, *there exists* $L(C) \in U_7^{20}:\ \frac{c_{0,0}{c_{1,3}}^7}{{c_{1,4}}^6} = \lambda$.

   Then orbits in U_7^{20} *can be parameterized as* $L\left(1, 0, 0, 0, \lambda, 1, 0\right),\ \lambda \in \mathbb{C}$.

Proof. Here, $B_1 = \frac{c_{0,0}}{{A_0}^3}$. □

Proposition 2.13.

The subsets $U_7^6,\ U_7^7,\ U_7^{11},\ U_7^{12},\ U_7^{15},\ U_7^{16},\ U_7^{18},\ U_7^{19},\ U_7^{21},\ U_7^{22},\ U_7^{23},\ U_7^{24},\ U_7^{25}$ *and* U_7^{26} *are single orbits with representatives* $L(1,0,0,0,0,0,0,1),\ L(0,0,0,0,0,0,0,1),$ $L(0,0,0,1,0,1,0),\ L(0,0,0,1,0,0,0),\ L(1,0,1,0,0,0,0),\ L(0,0,1,0,0,0,0),\ L(0,1,0,0,1,0,0),$ $L(0,1,0,0,0,0,0),\ L(1,0,0,0,1,0,0),\ L(1,0,0,0,0,0,0),\ L(0,0,0,0,1,1,0),\ L(0,0,0,0,0,1,0),$ $L(0,0,0,0,1,0,0)$ *and* $L(0,0,0,0,0,0,0)$*, respectively.*

Proof. To prove it, we give the respective values of A_0, A_1, B_1, B_2 and B_3 in the base change (8) leading to the appropriate representatives.

For U_7^6 and U_7^7 :

$$B_1 = \frac{{A_0}^2}{c_{2,3}} \text{ and } B_3 = \frac{{A_0}^4 c_{1,4} + {B_2}^2 {c_{2,3}}^3}{2\,{A_0}^2 {c_{2,3}}^2}.$$

For U_7^{11} and U_7^{12} :

$$A_1 = \frac{-A_0 c_{1,3}}{2\,{c_{1,2}}^2} \quad \text{and} \quad B_1 = \frac{{A_0}^2}{c_{1,2}}.$$

For U_7^{15} and U_7^{16} :

$$A_1 = \frac{-A_0 c_{0,1}}{2\,c_{1,1}} \quad \text{and} \quad B_1 = \frac{{A_0}^5}{c_{1,1}}.$$

For U_7^{18} :

$$A_1 = -\frac{A_0 c_{0,0}}{c_{0,1}} \quad \text{and} \quad B_1 = \frac{{A_0}^3}{c_{1,3}}.$$

For U_7^{19} :

$$A_1 = -\frac{A_0 c_{0,0}}{c_{0,1}}.$$

For U_7^{21} and U_7^{22} :

$$B_1 = \frac{c_{0,0}}{{A_0}^3}.$$

For U_7^{23} and U_7^{24} :

$$B_1 = \frac{{A_0}^4}{c_{1,4}}.$$

□

Note that the orbits U_7^6 and U_7^7 can be included in the parametric family of orbits U_7^3 and U_7^4, respectively at $\lambda = 0$.

2.2 Isomorphism criterion for $TLeib_8$

Here, we have $n = 7$, means odd case, and any algebra L from $TLeib_8$ can be represented as one dimensional central extension ($C(L) =< e_7 >$) of 7-dimensional filiform Lie algebra with adapted basis $\{e_0, e_1, ..., e_6\}$ (see Theorem 1.1) and on the adapted basis $\{e_0, e_1, ..., e_7\}$ the class $TLeib_8$ can be represented as follows:

$$TLeib_8 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le 6, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le 6, \\ [e_0, e_0] = b_{0,0} e_7, \\ [e_0, e_1] = -e_2 + b_{0,1} e_7, \\ [e_1, e_1] = b_{1,1} e_7, \\ [e_1, e_2] = -[e_2, e_1] = a_{1,4}\, e_4 + a_{1,5}\, e_5 + a_{1,6}\, e_6 + b_{1,2} e_7, \\ [e_1, e_3] = -[e_3, e_1] = a_{1,4}\, e_5 + a_{1,5}\, e_6 + b_{1,3} e_7, \\ [e_1, e_4] = -[e_4, e_1] = (a_{1,4} - a_{2,6})\, e_6 + b_{1,4} e_7, \\ [e_1, e_5] = -[e_5, e_1] = b_{1,5} e_7, \\ [e_2, e_3] = -[e_3, e_2] = a_{2,6} e_6 + b_{2,3} e_7, \\ [e_1, e_5] = -[e_5, e_1] = b_{1,5} e_6, \\ [e_2, e_4] = -[e_4, e_2] = b_{2,4} e_7, \\ [e_i, e_{7-i}] = -[e_{7-i}, e_i] = (-1)^i\, b_{3,4} e_7, \quad 1 \le i \le n-1. \end{cases}$$

The next lemma specifies the set of structure constants of algebras from $TLeib_8$.

Lemma 2.2. *The structure constants of algebras from $TLeib_8$ satisfy the following constraints:*

$$b_{1,3} = a_{1,6}, \quad b_{1,4} = a_{1,5} - b_{2,3}, \quad b_{2,4} = a_{2,6}, \quad b_{1,5} = a_{1,4} - 2a_{2,6}, \quad \textit{and} \quad b_{1,6}\,(a_{2,6} + 2\,a_{1,4}) = 0.$$

Proof. These relations come out from sequentially applications of the Leibniz identity to the triples of the basis vectors $\{e_0, e_1, e_2\}, \{e_0, e_1, e_3\}, \{e_0, e_2, e_3\}, \{e_0, e_1, e_4\}$ and $\{e_1, e_2, e_3\}$ respectively. $\square$

Further unifying the $TLeib_8$ table of multiplication we rewrite it via parameters $c_{0,0}$, $c_{0,1}$, $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, $c_{1,4}$, $c_{1,5}$, $c_{2,3}$, $c_{2,4}$ as follows:

$$TLeib_8 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le 6, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le 6, \\ [e_0, e_0] = c_{0,0} e_7, \\ [e_0, e_1] = -e_2 + c_{0,1} e_7, \\ [e_1, e_1] = c_{1,1} e_7, \\ [e_1, e_2] = -[e_2, e_1] = c_{1,2}\, e_4 + c_{1,3}\, e_5 + c_{1,4} e_6 + c_{1,5} e_7, \\ [e_1, e_3] = -[e_3, e_1] = c_{1,2}\, e_5 + c_{1,3} e_6 + c_{1,4} e_7, \\ [e_1, e_4] = -[e_4, e_1] = (c_{1,2} - c_{2,3})\, e_6 + (c_{1,3} - c_{2,4})\, e_7, \\ [e_1, e_5] = -[e_5, e_1] = (c_{1,2} - 2c_{2,3})\, e_7, \\ [e_2, e_3] = -[e_3, e_2] = c_{2,3}\, e_6 + c_{2,4}\, e_7, \\ [e_2, e_4] = -[e_4, e_2] = c_{2,3}\, e_7, \\ [e_i, e_{7-i}] = -[e_{7-i}, e_i] = (-1)^i\, c_{3,4} e_7, \quad 1 \le i \le n-1. \end{cases}$$

So an algebra from $TLeib_8$ with parameters $c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{2,3}, c_{2,4}, c_{3,4}$ is denoted by $L(C)$, where $C = (c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{2,3}, c_{2,4}, c_{3,4})$.

Note that $L(C) \in TLeib_8$ if and only if $c_{3,4}\,(c_{2,3} + 2\,c_{1,2}) = 0$.

Notice that according to the Proposition 1.2 and 1.3 the adapted transformation in $TLeib_8$ can be taken in the form

$$\begin{cases} f(e_0) = A_0\, e_0 + A_1\, e_1 + A_2\, e_2 + A_3\, e_3 + A_4\, e_4, \\ f(e_1) = B_1\, e_1 + B_2\, e_2 + B_3\, e_3 + B_4\, e_4 + B_5\, e_5, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \le i \le n-1, \end{cases}$$

where $A_0\, B_1 (A_0 + A_1 c_{3,4}) \neq 0$.

The next theorem represents the action of the adapted base change to the parameters $c_{0,0}, c_{0,1}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{2,3}, c_{2,4}, c_{3,4}$ of an algebra from $TLeib_8$.

Theorem 2.2. (Isomorphism criterion for $TLeib_8$) *Two filiform Leibniz algebras $L(C)$ and $L(C')$, where $C=(c_{0,0},c_{0,1},c_{1,1},c_{1,2},c_{1,3},c_{1,4},c_{1,5},c_{2,3},c_{2,4},c_{3,4})$ and $C'=(c'_{0,0},c'_{0,1},c'_{1,1},c'_{1,2},c'_{1,3},c'_{1,4},c'_{1,5},c'_{2,3},c'_{2,4},c'_{3,4})$, from $TLeib_8$ are isomorphic, if and only if there exist $A_0,A_1,A_2,A_3,A_4,B_1,B_2,B_3,B_4,B_5\in\mathbb{C}$, such that $A_0B_1(A_0+A_1c_{3,4})\neq 0$ and the following equalities hold:*

$$
\begin{aligned}
c'_{0,0} &= \frac{{A_0}^2c_{0,0}+A_0A_1c_{0,1}+{A_1}^2c_{1,1}}{c_1{A_0}^5\left(A_0+A_1c_{3,4}\right)},\\
c'_{0,1} &= \frac{A_0c_{0,1}+2\,A_1c_{1,1}}{{A_0}^5\left(A_0+A_1c_{3,4}\right)},\\
c'_{1,1} &= \frac{B_1c_{1,1}}{{A_0}^5\left(A_0+A_1c_{3,4}\right)},\\
c'_{1,2} &= \frac{B_1c_{1,2}}{{A_0}^2},\\
c'_{1,3} &= \frac{B_1\left(A_0c_{1,3}+2\,A_1{c_{1,2}}^2\right)}{{A_0}^4},\\
c'_{1,4} &= \frac{{A_0}^2\,{B_1}^2c_{1,4}+{A_0}^2({B_2}^2-2B_1\,B_3)\,c_{2,3}+A_1\,{B_1}^2(5\,c_{1,2}-c_{2,3})\,(A_0\,c_{1,3}-B_1\,{c_{1,2}}^2)}{{A_0}^6B_1},\\
c'_{1,5} &= -\frac{1}{{A_0}^8{B_1}^2\left(A_0+c_{3,4}A_1\right)}({A_0}^3A_1{B_1}^3c_{1,3}c_{2,4}-3\,{A_0}^3A_2{B_1}^2B_2c_{1,2}c_{2,3}-2\,{A_0}^3A_2{B_1}^2B_2c_{3,4}c_{1,3}+\\
&\quad {A_0}^3A_1B_1{B_2}^2c_{3,4}c_{1,3}-3\,A_0{A_1}^3{B_1}^3{c_{1,2}}^2{c_{2,3}}^2-3\,{A_0}^2{A_1}^2{B_1}^3c_{1,3}{c_{2,3}}^2-2\,{A_0}^4B_1B_2B_4c_{3,4}-\\
&\quad 6\,{A_0}^3A_1{B_1}^3c_{1,2}c_{1,4}-6\,{A_0}^2{A_1}^2{B_1}^3c_{1,2}c_{3,4}c_{1,4}+4\,{A_0}^2{A_1}^2{B_1}^2B_3c_{1,2}c_{3,4}c_{2,3}+\\
&\quad 12\,{A_0}^2{A_1}^2{B_1}^3c_{1,2}c_{1,3}c_{2,3}-{A_0}^2{A_2}^2{B_1}^3{c_{1,2}}^2c_{3,4}-2\,{A_0}^3A_1{B_2}^3c_{1,2}c_{3,4}+2\,{A_0}^3A_3{B_1}^3c_{1,3}c_{3,4}+\\
&\quad 6\,{A_0}^2{A_1}^2B_1{B_2}^2{c_{1,2}}^2c_{3,4}+2\,{A_0}^3A_2B_1{B_2}^2c_{1,2}c_{3,4}+2\,{A_0}^2A_1A_2{B_1}^2B_2{c_{1,2}}^2c_{3,4}-\\
&\quad 2\,{A_0}^3A_2{B_1}^2B_3c_{1,2}c_{3,4}-14\,{A_0}^2{A_1}^2{B_1}^2B_3{c_{1,2}}^2c_{3,4}+2\,{A_1}^4{B_1}^3{c_{1,2}}^3c_{3,4}c_{2,3}-\\
&\quad 2\,{A_0}^2{A_1}^2B_1{B_2}^2c_{2,3}c_{3,4}c_{1,2}+3\,{A_0}^3A_1{B_1}^3c_{1,4}c_{2,3}+2\,{A_0}^3A_4{B_1}^3c_{1,2}c_{3,4}+\\
&\quad 4\,{A_0}^3A_1B_1B_2B_3c_{1,2}c_{3,4}+9\,{A_0}^3A_1{B_1}^2B_3c_{1,2}c_{2,3}+3\,{A_0}^3A_3{B_1}^3c_{1,2}c_{2,3}-\\
&\quad 21\,{A_0}^2{A_1}^2{B_1}^3{c_{1,2}}^2c_{1,3}+{A_0}^2{A_1}^2{B_1}^3{c_{1,2}}^2c_{2,4}+{A_0}^4B_1{B_3}^2c_{3,4}-3\,{A_0}^3A_1{B_1}^3{c_{1,3}}^2-\\
&\quad 14\,A_0{A_1}^3{B_1}^3{c_{1,2}}^4-25\,{A_1}^4{B_1}^3{c_{1,2}}^4c_{3,4}-2\,{A_0}^3A_3{B_1}^2B_2c_{1,2}c_{3,4}-3\,{A_0}^3A_1B_1{B_2}^2c_{2,3}c_{1,2}+\\
&\quad 3\,{A_0}^3A_1B_1{B_2}^2{c_{2,3}}^2-2\,{A_0}^3A_1{B_1}^2B_4c_{1,2}c_{3,4}-6\,{A_0}^3A_1{B_1}^2B_3{c_{2,3}}^2-3\,{A_0}^4B_3B_1B_2c_{2,3}-\\
&\quad {A_0}^4{B_1}^3c_{1,5}+{A_0}^4{B_2}^3c_{2,3}-{A_0}^4B_1{B_2}^2c_{2,4}+3\,{A_0}^4{B_1}^2B_4c_{2,3}+2\,{A_0}^4{B_1}^2B_5c_{3,4}+\\
&\quad 2\,{A_0}^4B_3{B_1}^2c_{2,4}-2\,{A_0}^2{A_1}^2{B_1}^3{c_{1,3}}^2c_{3,4}-32\,A_0{A_1}^3{B_1}^3c_{1,3}c_{3,4}{c_{1,2}}^2+\\
&\quad 2\,A_0{A_1}^3{B_1}^3c_{1,3}c_{3,4}c_{1,2}c_{2,3}+9\,A_0{A_1}^3{B_1}^3{c_{1,2}}^3c_{2,3}),\\
c'_{2,3} &= \frac{B_1\,c_{2,3}}{A_0^2},\\
c'_{2,4} &= \frac{1}{A_0^4B_1\left(A_0+c_{3,4}A_1\right)}(A_0^2B_1^2c_{2,4}+\left(-A_0^2B_2^2+2\,A_0^2B_1B_3-A_0A_1B_1^2c_{1,3}-10\,A_1^2B_1^2c_{1,2}^2\right)c_{3,4}\\
&\quad +3\,A_0A_1B_1^2c_{2,3}c_{1,2}-3\,A_0A_1B_1^2c_{2,3}^2),\\
c'_{3,4} &= \frac{B_1c_{3,4}}{A_0+c_{3,4}A_1}.
\end{aligned}
$$

The following base change is adapted and it transforms $L(C)$ to $L(C')$

$$
\begin{aligned}
e'_0 &= A_0\, e_0 + A_1\, e_1 + A_2\, e_2 + A_3\, e_3 + A_4\, e_4,\\
e'_1 &= B_1\, e_1 + B_2\, e_2 + B_3\, e_3 + B_4\, e_4 + B_5\, e_5,\\
e'_2 &= A_0B_1\, e_2 + A_0B_2e_3 + (A_0B_3 + (A_2B_1 - A_1B_2)c_{1,2})\, e_4 + (A_0B_4 + (A_3B_1 - A_1B_3)c_{1,2} +\\
&\quad (A_2B_1 - A_1B_2)c_{1,3})e_5 + ((A_3B_2 - A_4B_1 - A_2B_3 + A_1B_4)\, c_{2,3} + (A_2B_1 - A_1B_2)\, c_{1,4} +\\
&\quad (A_3B_1 - A_1B_3)\, c_{1,3} + (-A_1B_4 + A_4B_1)\, c_{1,2} + A_0B_5)e_6 + (*)e_7,\\
e'_3 &= A_0^2B_1\, e_3 + \left(A_0^2B_2 - A_0A_1B_1c_{1,2}\right) e_4 + (A_0^2B_3 - A_1A_0B_1c_{1,3} + A_0\,(-2\,A_1B_2 + A_2B_1)\, c_{1,2})e_5\\
&\quad +(\left(-A_1A_2B_1 + {A_1}^2B_2\right) {c_{1,2}}^2 + \left(\left(A_1A_2B_1 - {A_1}^2B_2\right) c_{2,3} + A_3A_0B_1 - 2\,A_1A_0B_3\right) c_{1,2} + \qquad (9)\\
&\quad (-2\,A_1A_0B_2 + A_0A_2B_1)\, c_{1,3} + (A_3A_0B_1 - A_2A_0B_2 + A_1A_0B_3)\, c_{2,3} + A_1A_0B_1c_{1,4} +\\
&\quad {A_0}^2B_4)e_6 + (*)e_7,\\
e'_4 &= A_0^3B_1\, e_4 + (A_0^3B_2 - 2\,A_0^2A_1B_1c_{1,2})e_5 + A_0({A_0}^2B_3 - 2\,A_1A_0B_1c_{1,3} - A_0A_2B_1c_{2,3} -\\
&\quad 3\,A_1A_0B_2c_{1,2} + A_0A_1B_2c_{2,3} + A_0A_2B_1c_{1,2} + {A_1}^2B_1{c_{1,2}}^2 - {A_1}^2B_1c_{1,2}c_{2,3})e_6 + (*)e_7,\\
e'_5 &= A_0^4\, B_1\, e_5 + (A_0^4B_2 - 3\,A_1A_0^3B_1c_{1,2} + A_1A_0^3B_1c_{2,3})e_6 + (*)e_7,\\
e'_6 &= A_0^5\, B_1\, e_6 + (*)e_7,\\
e'_7 &= B_1{A_0}^5\,(A_0 + A_1c_{3,4})\, e_7.
\end{aligned}
$$

The next section deals with the applications of the results of the previous section to the classification problem of $TLeib_8$.

2.2.1 Isomorphism classes of $TLeib_8$

In this section we give a list of all algebras from $TLeib_8$;
Represent $TLeib_8$ as a union of the following subsets:

$M_8^1 = \{L(C) \in TLeib_8\ : c_{3,4} \neq 0\}$,

$U_8^1 = \{L(C) \in TLeib_8\ : c_{1,1} \neq 0\}$;
$U_8^2 = \{L(C) \in TLeib_8\ : c_{1,1} = 0,\ c_{0,1} \neq 0\}$;
$U_8^3 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = 0,\ c_{1,2} \neq 0\}$;
$U_8^4 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = 0,\ c_{1,3} \neq 0,\ c_{1,4} \neq 0\}$;
$U_8^5 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = 0,\ c_{1,3} \neq 0,\ c_{1,4} = 0,\ c_{0,0} \neq 0\}$;
$U_8^6 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = 0,\ c_{1,3} \neq 0,\ c_{1,4} = c_{0,0} = 0\}$;
$U_8^7 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = c_{1,3} = 0,\ c_{1,4} \neq 0,\ c_{0,0} \neq 0\}$;
$U_8^8 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = c_{1,3} = 0,\ c_{1,4} \neq 0,\ c_{0,0} = 0\}$;
$U_8^9 = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = c_{1,3} = c_{1,4} = 0,\ c_{0,0} \neq 0\}$;
$U_8^{10} = \{L(C) \in TLeib_8\ : c_{1,1} = c_{0,1} = c_{1,2} = c_{1,3} = c_{1,4} = 0,\ c_{0,0} = 0\}$;

$M_8^2 = \{L(C) \in TLeib_8\ : c_{3,4} = 0\}$

$U_8^{11} = \{L(C) \in TLeib_8\ : c_{2,3} \neq 0,\ c_{1,1} \neq 0\}$;
$U_8^{12} = \{L(C) \in TLeib_8\ : c_{2,3} \neq 0,\ c_{1,1} = 0,\ c_{0,1} \neq 0\}$;
$U_8^{13} = \{L(C) \in TLeib_8\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = 0,\ c_{1,2} \neq 0,$
$2\,c_{2,4}c_{1,2}^2 - 3\,c_{1,3}c_{2,3}c_{1,2} + 3\,c_{1,3}c_{2,3}^2 \neq 0\}$;
$U_8^{14} = \{L(C) \in TLeib_8\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = 0,\ c_{1,2} \neq 0,$
$2\,c_{2,4}c_{1,2}^2 - 3\,c_{1,3}c_{2,3}c_{1,2} + 3\,c_{1,3}c_{2,3}^2 = 0,\ c_{0,0} \neq 0\}$;
$U_8^{15} = \{L(C) \in TLeib_8\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = 0,\ c_{1,2} \neq 0,$
$2\,c_{2,4}{c_{1,2}}^2 - 3\,c_{1,3}c_{2,3}c_{1,2} + 3\,c_{1,3}{c_{2,3}}^2 = 0,\ c_{0,0} = 0\}$;
$U_8^{16} = \{L(C) \in TLeib_8\ : c_{2,3} \neq 0,\ c_{1,1} = c_{0,1} = c_{1,2} = 0,\ c_{1,3} \neq 0,\ c_{0,0} \neq 0\}$;

$U_8^{17}=\{L(C)\in TLeib_8\ :c_{2,3}\neq 0,\ c_{1,1}=c_{0,1}=c_{1,2}=c_{1,3}=0,\ c_{0,0}\neq 0\};$
$U_8^{18}=\{L(C)\in TLeib_8\ :c_{2,3}\neq 0,\ c_{1,1}=c_{0,1}=c_{1,2}=c_{1,3}=c_{0,0}=0\};$
$U_8^{19}=\{L(C)\in TLeib_8\ :c_{2,3}=0,\ c_{2,4}\neq 0,\ c_{1,2}\neq 0\};$
$U_8^{20}=\{L(C)\in TLeib_8\ :c_{2,3}=0,\ c_{2,4}\neq 0,\ c_{1,2}=0,\ c_{1,1}\neq 0\};$
$U_8^{21}=\{L(C)\in TLeib_8\ :c_{2,3}=0,\ c_{2,4}\neq 0,\ c_{1,2}=c_{1,1}=0,\ c_{0,1}\neq 0\};$
$U_8^{22}=\{L(C)\in TLeib_8\ :c_{2,3}=0,\ c_{2,4}\neq 0,\ c_{1,2}=c_{1,1}=c_{0,1}=0,\ c_{1,4}\neq 0\};$
$U_8^{23}=\{L(C)\in TLeib_8\ :c_{2,3}=0,\ c_{2,4}\neq 0,\ c_{1,2}=c_{1,1}=c_{0,1}=c_{1,4}=0,\ c_{0,0}\neq 0\};$
$U_8^{24}=\{L(C)\in TLeib_8\ :c_{2,3}=0,\ c_{2,4}\neq 0,\ c_{1,2}=c_{1,1}=c_{0,1}=c_{1,4}=c_{0,0}=0\};$
$U_8^{25}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=0,\ c_{1,2}\neq 0,\ 4\,c_{1,4}c_{1,2}-5\,c_{1,3}^2\neq 0\};$
$U_8^{26}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=0,\ c_{1,2}\neq 0,\ 4\,c_{1,4}c_{1,2}-5\,c_{1,3}^2=0,\ -7\,c_{1,3}^3+4\,c_{1,5}c_{1,2}^2\neq 0\};$
$U_8^{27}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=0,\ c_{1,2}\neq 0,\ 4\,c_{1,4}c_{1,2}-5\,c_{1,3}^2=0,$
$-7\,c_{1,3}^3+4\,c_{1,5}c_{1,2}^2=0,\ c_{1,1}\neq 0\};$
$U_8^{28}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=0,\ c_{1,2}\neq 0,\ 4\,c_{1,4}c_{1,2}-5\,c_{1,3}^2=0,$
$-7\,c_{1,3}^3+4\,c_{1,5}c_{1,2}^2=0,\ c_{1,1}=0,\ c_{0,1}\neq 0\};$
$U_8^{29}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=0,\ c_{1,2}\neq 0,\ 4\,c_{1,4}c_{1,2}-5\,c_{1,3}^2=0,$
$-7\,c_{1,3}^3+4\,c_{1,5}c_{1,2}^2=0,\ c_{1,1}=c_{0,1}=0,\ c_{0,0}\neq 0\};$
$U_8^{30}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=0,\ c_{1,2}\neq 0,\ 4\,c_{1,4}c_{1,2}-5\,c_{1,3}^2=0,$
$-7\,c_{1,3}^3+4\,c_{1,5}c_{1,2}^2=0,\ c_{1,1}=c_{0,1}=c_{0,0}=0\};$
$U_8^{31}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=0,\ c_{1,3}\neq 0,\ c_{1,4}\neq 0\};$
$U_8^{32}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=0,\ c_{1,3}\neq 0,\ c_{1,4}=0,\ c_{1,1}\neq 0\};$
$U_8^{33}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=0,\ c_{1,3}\neq 0,\ c_{1,4}=c_{1,1}=0,\ c_{0,1}\neq 0\};$
$U_8^{34}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=0,\ c_{1,3}\neq 0,\ c_{1,4}=c_{1,1}=c_{0,1}=0,\ c_{0,0}\neq 0\};$
$U_8^{35}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=0,\ c_{1,3}\neq 0,\ c_{1,4}=c_{1,1}=c_{0,1}=c_{0,0}=0\};$
$U_8^{36}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=0,\ c_{1,1}\neq 0,\ c_{1,5}\neq 0\};$
$U_8^{37}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=0,\ c_{1,1}\neq 0,\ c_{1,5}=0,\ c_{1,4}\neq 0\};$
$U_8^{38}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=0,\ c_{1,1}\neq 0,\ c_{1,5}=c_{1,4}=0,$
$4\,c_{0,0}c_{1,1}-c_{0,1}^2\neq 0\};$
$U_8^{39}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=0, c_{1,1}\neq 0,\ c_{1,5}=c_{1,4}=0,$
$4\,c_{0,0}c_{1,1}-c_{0,1}^2=0\};$
$U_8^{40}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=0,\ c_{0,1}\neq 0,\ c_{1,5}\neq 0\};$
$U_8^{41}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=0,\ c_{0,1}\neq 0,\ c_{1,5}=0,\ c_{1,4}\neq 0\};$
$U_8^{42}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=0,\ c_{0,1}\neq 0,\ c_{1,5}=c_{1,4}=0\};$
$U_8^{43}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=0, c_{1,4}\neq 0,\ c_{1,5}\neq 0\};$
$U_8^{44}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=0,\ c_{1,4}\neq 0,\ c_{1,5}=0,\ c_{0,0}\neq 0\};$
$U_8^{45}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=0,\ c_{1,4}\neq 0,\ c_{1,5}=c_{0,0}=0\};$
$U_8^{46}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=c_{1,4}=0,\ c_{1,5}\neq 0,\ c_{0,0}\neq 0\};$
$U_8^{47}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=c_{1,4}=0,\ c_{1,5}\neq 0,\ c_{0,0}=0\};$
$U_8^{48}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=c_{1,4}=c_{1,5}=0,\ c_{0,0}\neq 0\};$
$U_8^{49}=\{L(C)\in TLeib_8\ :c_{2,3}=c_{2,4}=c_{1,2}=c_{1,3}=c_{1,1}=c_{0,1}=c_{1,4}=c_{1,5}=c_{0,0}=0\}.$

Proposition 2.14.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^1 are isomorphic, if and only if*

$$\frac{\left(4\,c'_{0,0}c'_{1,1}-{c'}_{0,1}^{2}\right){c'}_{3,4}^{2}}{\left(-2\,c'_{1,1}+c'_{0,1}c'_{3,4}\right)^2}=\frac{\left(4\,c_{0,0}c_{1,1}-c_{0,1}^2\right)c_{3,4}^2}{\left(-2\,c_{1,1}+c_{0,1}c_{3,4}\right)^2},\quad \frac{\left(-2\,c'_{1,1}+c'_{0,1}c'_{3,4}\right)^5{c'}_{1,2}^{5}}{{c'}_{3,4}^{4}{c'}_{1,1}^{6}}=\frac{\left(-2\,c_{1,1}+c_{0,1}c_{3,4}\right)^5c_{1,2}^5}{c_{3,4}^4c_{1,1}^6},$$

$$\frac{\left(-2\,c'_{1,1}+c'_{0,1}c'_{3,4}\right)^5\left(-c'_{1,3}c'_{1,1}+{c'}_{1,2}^{2}c'_{0,1}\right)^5}{{c'}_{3,4}^{3}{c'}_{1,1}^{12}}=\frac{\left(-2\,c_{1,1}+c_{0,1}c_{3,4}\right)^5\left(-c_{1,3}c_{1,1}+c_{1,2}^2c_{0,1}\right)^5}{c_{3,4}^3c_{1,1}^{12}},$$

$$\frac{\left(-2\,c'_{1,1}+c'_{0,1}c'_{3,4}\right)^5}{{c'}_{1,1}^{18}{c'}_{3,4}^{7}}(27\,c'_{3,4}{c'}_{1,2}^{3}{c'}_{0,1}^{2}-18\,c'_{3,4}c'_{1,2}c'_{0,1}c'_{1,3}c'_{1,1}+4\,c'_{3,4}c'_{1,4}{c'}_{1,1}^{2}-72\,{c'}_{1,2}^{3}c'_{0,1}c'_{1,1}-$$

$$8\,c'_{1,2}c'_{2,4}{c'}_{1,1}^{2})^5=\frac{\left(-2\,c_{1,1}+c_{0,1}c_{3,4}\right)^5}{c_{1,1}^{18}c_{3,4}^{7}}(27\,c_{3,4}c_{1,2}^3c_{0,1}^2-18\,c_{3,4}c_{1,2}c_{0,1}c_{1,3}c_{1,1}+4\,c_{3,4}c_{1,4}c_{1,1}^2$$

$$-72\,c_{1,2}^3c_{0,1}c_{1,1}-8\,c_{1,2}c_{2,4}c_{1,1}^2)^5.$$

2. *For any* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, *there exists* $L(C) \in U_8^1$: $\dfrac{\left(4\, c_{0,0}c_{1,1} - c_{0,1}^2\right) c_{3,4}^2}{\left(-2\, c_{1,1} + c_{0,1}c_{3,4}\right)^2} = \lambda_1$,
$\dfrac{\left(-2\, c'_{1,1} + c'_{0,1}c'_{3,4}\right)^5 {c'}_{1,2}^5}{{c'}_{3,4}^4 {c'}_{1,1}^6} = \lambda_2$, $\quad \dfrac{\left(-2\, c_{1,1} + c_{0,1}c_{3,4}\right)^5 \left(-c_{1,3}c_{1,1} + c_{1,2}^2 c_{0,1}\right)^5}{c_{3,4}^3 c_{1,1}^{12}} = \lambda_3$,
$\dfrac{\left(-2\, c_{1,1} + c_{0,1}c_{3,4}\right)^5}{c_{1,1}^{18} c_{3,4}^7}(27\, c_{3,4}c_{1,2}^3 c_{0,1}^2 - 18\, c_{3,4}c_{1,2}c_{0,1}c_{1,3}c_{1,1} + 4\, c_{3,4}c_{1,4}c_{1,1}^2 - 72\, c_{1,2}^3 c_{0,1}c_{1,1}$
$- 8\, c_{1,2}c_{2,4}c_{1,1}^2)^5 = \lambda_4$.
Then orbits in U_8^1 *can be parameterized as* $L\left(\lambda_1, 0, 1, \lambda_2, \lambda_3, \lambda_4, 0, -32\lambda_2, 0, 1\right),\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$.

Proof. See proof of Proposition 2.1, where we put $A_1 = \dfrac{-A_0 c_{0,1}}{2\, c_{1,1}}$, $B_1 = \dfrac{-A_0 \left(-2\, c_{1,1} + c_{0,1}c_{3,4}\right)}{2\, c_{3,4}c_{1,1}}$, and
$B_3 = -\dfrac{1}{2\, A_0 {c_{1,1}}^3 {c_{3,4}}^2 \left(-2\, c_{1,1} + c_{0,1}c_{3,4}\right)}(-8\, {A_0}^2 c_{2,4}{c_{1,1}}^4 + 8\, {A_0}^2 c_{2,4}c_{0,1}c_{3,4}{c_{1,1}}^3 - 2\, {A_0}^2 c_{2,4}{c_{0,1}}^2 {c_{3,4}}^2 {c_{1,1}}^2 -$
$72\, {A_0}^2 {c_{1,2}}^2 c_{0,1}{c_{1,1}}^3 + 92\, {A_0}^2 {c_{1,2}}^2 {c_{0,1}}^2 c_{3,4}{c_{1,1}}^2 - 38\, {A_0}^2 {c_{1,2}}^2 {c_{0,1}}^3 {c_{3,4}}^2 c_{1,1} - 4\, {A_0}^2 c_{0,1}c_{1,3}c_{3,4}{c_{1,1}}^3 +$
$4\, {A_0}^2 {c_{0,1}}^2 c_{1,3}{c_{3,4}}^2 {c_{1,1}}^2 - {A_0}^2 {c_{0,1}}^3 c_{1,3}{c_{3,4}}^3 c_{1,1} + 8\, {B_2}^2 {c_{3,4}}^3 {c_{1,1}}^4 + 5\, {A_0}^2 {c_{1,2}}^2 {c_{0,1}}^4 {c_{3,4}}^3)$, in base change (9). □

Proposition 2.15.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^2 *are isomorphic, if and only if*

$$\frac{\left(-c'_{0,1} + c'_{0,0}c'_{3,4}\right)^6 {c'}_{1,2}^5}{{c'}_{3,4}^5 {c'}_{0,1}^7} = \frac{\left(-c_{0,1} + c_{0,0}c_{3,4}\right)^6 c_{1,2}^5}{c_{3,4}^5 c_{0,1}^7},$$

$$\frac{\left(-c'_{0,1} + c'_{0,0}c'_{3,4}\right)^7 \left(-c'_{1,3}c'_{0,1} + 2\, {c'}_{1,2}^2 c'_{0,0}\right)^5}{{c'}_{3,4}^5 {c'}_{0,1}^{14}} = \frac{\left(-c_{0,1} + c_{0,0}c_{3,4}\right)^7 \left(-c_{1,3}c_{0,1} + 2\, c_{1,2}^2 c_{0,0}\right)^5}{c_{3,4}^5 c_{0,1}^{14}},$$

$\dfrac{\left(-c'_{0,1} + c'_{0,0}c'_{3,4}\right)^8}{{c'}_{0,1}^{21} {c'}_{3,4}^{10}}(27\, c'_{3,4}{c'}_{1,2}^3 {c'}_{0,0}^2 - 9\, c'_{3,4}c'_{1,2}c'_{0,0}c'_{1,3}c'_{0,1} + c'_{3,4}c'_{1,4}{c'}_{0,1}^2 - 36\, {c'}_{1,2}^3 c'_{0,0}c'_{0,1} -$
$2\, c'_{1,2}c'_{2,4}{c'}_{0,1}^2)^5 = \dfrac{\left(-c_{0,1} + c_{0,0}c_{3,4}\right)^8}{c_{0,1}^{21} c_{3,4}^{10}}(27\, c_{3,4}c_{1,2}^3 c_{0,0}^2 - 9\, c_{3,4}c_{1,2}c_{0,0}c_{1,3}c_{0,1} + c_{3,4}c_{1,4}c_{0,1}^2 -$
$36\, c_{1,2}^3 c_{0,0}c_{0,1} - 2\, c_{1,2}c_{2,4}c_{0,1}^2)^5$.

2. *For any* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, *there exists* $L(C) \in U_8^2$: $\dfrac{\left(-c_{0,1} + c_{0,0}c_{3,4}\right)^6 c_{1,2}^5}{c_{3,4}^5 c_{0,1}^7} = \lambda_1$,
$\dfrac{\left(-c_{0,1} + c_{0,0}c_{3,4}\right)^7 \left(-c_{1,3}c_{0,1} + 2\, c_{1,2}^2 c_{0,0}\right)^5}{c_{3,4}^5 c_{0,1}^{14}} = \lambda_2$, $\quad \dfrac{\left(-c_{0,1} + c_{0,0}c_{3,4}\right)^8}{c_{0,1}^{21} c_{3,4}^{10}}(27\, c_{3,4}c_{1,2}^3 c_{0,0}^2 -$
$9\, c_{3,4}c_{1,2}c_{0,0}c_{1,3}c_{0,1} + c_{3,4}c_{1,4}c_{0,1}^2 - 36\, c_{1,2}^3 c_{0,0}c_{0,1} - 2\, c_{1,2}c_{2,4}c_{0,1}^2)^5 = \lambda_3$.
Then orbits in U_8^2 *can be parameterized as* $L\left(0, 1, 0, \lambda_1, \lambda_2, \lambda_3, 0, -32\lambda_1, 0, 1\right),\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

Proof. The respective value of coefficients in the base change (9) are:

$$A_1 = -\frac{A_0 c_{0,0}}{c_{0,1}}, \quad B_1 = -\frac{A_0 \left(-c_{0,1} + c_{0,0}c_{3,4}\right)}{c_{3,4}c_{0,1}},$$

and
$B_3 = -\dfrac{1}{2\, A_0 {c_{0,1}}^3 {c_{3,4}}^2 \left(-c_{0,1} + c_{0,0}c_{3,4}\right)}(-{A_0}^2 c_{2,4}{c_{0,1}}^4 + 2\, {A_0}^2 c_{2,4}c_{0,0}c_{3,4}{c_{0,1}}^3 - {A_0}^2 c_{2,4}{c_{0,0}}^2 {c_{3,4}}^2 {c_{0,1}}^2 -$
$18\, {A_0}^2 {c_{1,2}}^2 c_{0,0}{c_{0,1}}^3 + 46\, {A_0}^2 {c_{1,2}}^2 {c_{0,0}}^2 c_{3,4}{c_{0,1}}^2 - 38\, {A_0}^2 {c_{1,2}}^2 {c_{0,0}}^3 {c_{3,4}}^2 c_{0,1} - {A_0}^2 c_{0,0}c_{1,3}c_{3,4}{c_{0,1}}^3 +$
$2\, {A_0}^2 {c_{0,0}}^2 c_{1,3}{c_{3,4}}^2 {c_{0,1}}^2 - {A_0}^2 {c_{0,0}}^3 c_{1,3}{c_{3,4}}^3 c_{0,1} + {B_2}^2 {c_{3,4}}^3 {c_{0,1}}^4 + 10\, {A_0}^2 {c_{1,2}}^2 {c_{0,0}}^4 {c_{3,4}}^3)$. □

Proposition 2.16.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^3 are isomorphic, if and only if*

$$\frac{c'_{0,0}{c'}_{1,2}^{9}{c'}_{3,4}^{6}}{\left(-2\,{c'}_{1,2}^{2}+c'_{1,3}c'_{3,4}\right)^{7}}=\frac{c_{0,0}c_{1,2}^{9}c_{3,4}^{6}}{\left(-2\,c_{1,2}^{2}+c_{1,3}c_{3,4}\right)^{7}},$$

$$\frac{c'_{3,4}}{\left(-2\,b_{1,2}^{2}+c'_{1,3}c'_{3,4}\right)^{2}}\left(9\,c'_{3,4}{c'}_{1,3}^{2}+4\,c'_{3,4}c'_{1,4}c'_{1,2}-72\,{c'}_{1,2}^{2}c'_{1,3}-8\,{c'}_{1,2}^{2}c'_{2,4}\right)=$$

$$\frac{c_{3,4}}{\left(-2\,c_{1,2}^{2}+c_{1,3}c_{3,4}\right)^{2}}\left(9\,c_{3,4}c_{1,3}^{2}+4\,c_{3,4}c_{1,4}c_{1,2}-72\,c_{1,2}^{2}c_{1,3}-8\,c_{1,2}^{2}c_{2,4}\right).$$

2. *For any $\lambda_1, \lambda_2 \in \mathbb{C}$, there exists $L(C) \in U_8^3$:* $\dfrac{c_{0,0}c_{1,2}^{9}c_{3,4}^{6}}{\left(-2\,c_{1,2}^{2}+c_{1,3}c_{3,4}\right)^{7}}=\lambda_1,$

$$\frac{c_{3,4}}{\left(-2\,c_{1,2}^{2}+c_{1,3}c_{3,4}\right)^{2}}\left(9\,c_{3,4}c_{1,3}^{2}+4\,c_{3,4}c_{1,4}c_{1,2}-72\,c_{1,2}^{2}c_{1,3}-8\,c_{1,2}^{2}c_{2,4}\right)=\lambda_2.$$

Then orbits in U_8^3 can be parameterized as $L\left(\lambda_1,0,0,1,0,\lambda_2,0,-2,0,1\right),\ \lambda_1,\lambda_2\in\mathbb{C}.$

Proof. Here,

$$A_0=\frac{2\,{c_{1,2}}^{2}-c_{1,3}c_{3,4}}{2\,c_{3,4}c_{1,2}},\ A_1=-\frac{\left(2\,{c_{1,2}}^{2}-c_{1,3}c_{3,4}\right)c_{1,3}}{4\,c_{3,4}{c_{1,2}}^{3}},\ B_1=\frac{\left(2\,{c_{1,2}}^{2}-c_{1,3}c_{3,4}\right)^{2}}{4\,{c_{3,4}}^{2}{c_{1,2}}^{3}},$$

and

$$B_3=\frac{1}{8\,{c_{3,4}}^{3}{c_{1,2}}^{5}\left(2\,{c_{1,2}}^{2}-c_{1,3}c_{3,4}\right)^{2}}(-16\,c_{2,4}{c_{1,2}}^{10}+32\,c_{2,4}{c_{1,2}}^{8}c_{1,3}c_{3,4}-24\,c_{2,4}{c_{1,2}}^{6}{c_{1,3}}^{2}{c_{3,4}}^{2}+8\,c_{2,4}{c_{1,3}}^{3}{c_{3,4}}^{3}{c_{1,2}}^{4}-c_{2,4}{c_{1,3}}^{4}{c_{3,4}}^{4}{c_{1,2}}^{2}-144\,c_{1,3}{c_{1,2}}^{10}+320\,{c_{1,3}}^{2}{c_{1,2}}^{8}c_{3,4}-280\,{c_{1,3}}^{3}{c_{1,2}}^{6}{c_{3,4}}^{2}+120\,{c_{1,3}}^{4}{c_{3,4}}^{3}{c_{1,2}}^{4}-25\,{c_{1,3}}^{5}{c_{3,4}}^{4}{c_{1,2}}^{2}+2\,{c_{1,3}}^{6}{c_{3,4}}^{5}+16\,{B_2}^{2}{c_{3,4}}^{5}{c_{1,2}}^{8}).$$ □

Proposition 2.17.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^4 are isomorphic, if and only if* $\dfrac{{c'}_{1,3}^{11}c'_{0,0}}{{c'}_{1,4}^{9}c'_{3,4}}=\dfrac{c_{1,3}^{11}c_{0,0}}{c_{1,4}^{9}c_{3,4}}.$

2. *For any $\lambda \in \mathbb{C}$, there exists $L(C) \in U_8^4$:* $\dfrac{c_{1,3}^{11}c_{0,0}}{c_{1,4}^{9}c_{3,4}}=\lambda.$

Then orbits in U_8^4 can be parameterized as $L\left(\lambda,0,0,0,1,1,0,0,0,1\right),\ \lambda\in\mathbb{C}.$

Proof. For this case, we put in the base change (9) the following coefficients,

$$A_0=\frac{c_{1,4}}{c_{1,3}},\ B_1=\frac{c_{1,4}+A_1c_{3,4}c_{1,3}}{c_{3,4}c_{1,3}},$$

and

$$B_3=\frac{1}{2\,c_{1,3}{c_{1,4}}^{2}{c_{3,4}}^{2}\left(c_{1,4}+A_1c_{3,4}c_{1,3}\right)}(-{c_{1,4}}^{4}c_{2,4}-2\,A_1{c_{1,4}}^{3}c_{2,4}c_{3,4}c_{1,3}-{A_1}^{2}{c_{1,4}}^{2}c_{2,4}{c_{3,4}}^{2}{c_{1,3}}^{2}+18\,A_1{c_{1,2}}^{2}{c_{1,4}}^{3}c_{1,3}+46\,{A_1}^{2}{c_{1,2}}^{2}{c_{1,4}}^{2}c_{3,4}{c_{1,3}}^{2}+38\,{A_1}^{3}{c_{1,2}}^{2}c_{1,4}{c_{3,4}}^{2}{c_{1,3}}^{3}+A_1{c_{1,4}}^{3}c_{3,4}{c_{1,3}}^{2}+2\,{A_1}^{2}{c_{1,4}}^{2}{c_{3,4}}^{2}{c_{1,3}}^{3}+{A_1}^{3}c_{1,4}{c_{3,4}}^{3}{c_{1,3}}^{4}+{B_2}^{2}{c_{1,4}}^{2}{c_{3,4}}^{3}{c_{1,3}}^{2}+10\,{A_1}^{4}{c_{1,2}}^{2}{c_{3,4}}^{3}{c_{1,3}}^{4}).$$ □

Proposition 2.18.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{11} are isomorphic, if and only if*

$$\frac{\left(4\,c'_{0,0}c'_{1,1}-{c'}^2_{0,1}\right)^2{c'}^5_{2,3}}{{c'}^5_{1,1}}=\frac{\left(4\,c_{0,0}c_{1,1}-c^2_{0,1}\right)^2c^5_{2,3}}{c^5_{1,1}},$$

$$\frac{c'_{1,2}}{c'_{2,3}}=\frac{c_{1,2}}{c_{2,3}},\quad \frac{\left(-c'_{1,3}c'_{1,1}+{c'}^2_{1,2}c'_{0,1}\right)^4}{{c'}^3_{2,3}{c'}^5_{1,1}}=\frac{\left(-c_{1,3}c_{1,1}+c^2_{1,2}c_{0,1}\right)^4}{c^3_{2,3}c^5_{1,1}},$$

$$\frac{\left(2\,c'_{2,4}c'_{1,1}-3\,c'_{2,3}c'_{0,1}c'_{1,2}+3\,{c'}^2_{2,3}c'_{0,1}\right)^4}{{c'}^3_{2,3}{c'}^5_{1,1}}=\frac{\left(2\,c_{2,4}c_{1,1}-3\,c_{2,3}c_{0,1}c_{1,2}+3\,c^2_{2,3}c_{0,1}\right)^4}{c^3_{2,3}c^5_{1,1}}.$$

2. *For any $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{C}$, there exists $L(C)\in U_8^{11}$:*

$$\frac{\left(4\,c_{0,0}c_{1,1}-c^2_{0,1}\right)^2c^5_{2,3}}{c^5_{1,1}}=\lambda_1,\quad \frac{c_{1,2}}{c_{2,3}}=\lambda_2,$$

$$\frac{\left(-c_{1,3}c_{1,1}+c^2_{1,2}c_{0,1}\right)^4}{c^3_{2,3}c^5_{1,1}}=\lambda_3,\quad \frac{\left(2\,c_{2,4}c_{1,1}-3\,c_{2,3}c_{0,1}c_{1,2}+3\,c^2_{2,3}c_{0,1}\right)^4}{c^3_{2,3}c^5_{1,1}}=\lambda_4.$$

Then orbits in U_8^{11} can be parameterized as $L\left(\lambda_1,0,1,\lambda_2,\lambda_3,0,0,1,\lambda_4,0\right),\quad \lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{C}$.

Proof. For this case we put, $A_1=\dfrac{-A_0c_{0,1}}{2\,c_{1,1}}$, $B_1=\dfrac{{A_0}^2}{c_{2,3}}$, $B_3=\dfrac{1}{8\,{A_0}^2{c_{2,3}}^2{c_{1,1}}^2}(4\,{B_2}^2{c_{2,3}}^3{c_{1,1}}^2+4\,{A_0}^4c_{1,4}{c_{1,1}}^2+2\,{A_0}^4c_{0,1}c_{1,3}c_{2,3}c_{1,1}-10\,{A_0}^4c_{1,2}c_{0,1}c_{1,3}c_{1,1}+5\,{A_0}^4{c_{1,2}}^3{c_{0,1}}^2-{A_0}^4{c_{1,2}}^2{c_{0,1}}^2c_{2,3})$ and $B_4=\dfrac{1}{48\,{A_0}^4{c_{2,3}}^3{c_{1,1}}^3}(8\,{B_2}^3{c_{2,3}}^5{c_{1,1}}^3-20\,{A_0}^6c_{2,4}{c_{1,2}}^3{c_{0,1}}^2c_{1,1}+16\,{A_0}^6c_{1,5}c_{2,3}{c_{1,1}}^3-16\,{A_0}^6c_{2,4}c_{1,4}{c_{1,1}}^3-24\,{A_0}^6c_{0,1}{c_{1,3}}^2c_{2,3}{c_{1,1}}^2+17\,{A_0}^6{c_{0,1}}^3{c_{1,2}}^4c_{2,3}+12\,{B_2}^2{c_{2,3}}^4{A_0}^2c_{0,1}c_{1,2}{c_{1,1}}^2-48\,A_3{A_0}^5c_{1,2}{c_{2,3}}^2{c_{1,1}}^3+48\,B_2A_2{A_0}^3c_{1,2}{c_{2,3}}^3{c_{1,1}}^3-21\,{A_0}^6{c_{0,1}}^3{c_{1,2}}^3{c_{2,3}}^2-12\,{A_0}^6c_{1,2}c_{0,1}c_{1,4}c_{2,3}{c_{1,1}}^2-6\,{A_0}^6{c_{1,2}}^2{c_{0,1}}^2c_{1,3}c_{2,3}c_{1,1}+30\,{A_0}^6c_{1,2}{c_{0,1}}^2c_{1,3}{c_{2,3}}^2c_{1,1}+40\,{A_0}^6c_{2,4}c_{1,2}c_{0,1}c_{1,3}{c_{1,1}}^2+30\,B_2{A_0}^4{c_{1,2}}^3{c_{0,1}}^2{c_{2,3}}^2c_{1,1}+24\,B_2{A_0}^4c_{1,4}{c_{2,3}}^2{c_{1,1}}^3+12\,B_2{A_0}^4c_{0,1}c_{1,3}{c_{2,3}}^3{c_{1,1}}^2-60\,B_2{A_0}^4c_{1,2}c_{0,1}c_{1,3}{c_{2,3}}^2{c_{1,1}}^2-6\,B_2{A_0}^4{c_{1,2}}^2{c_{0,1}}^2{c_{2,3}}^3c_{1,1})$. $\square$

Proposition 2.19.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{12} are isomorphic, if and only if*

$$\frac{c'_{1,2}}{c'_{2,3}}=\frac{c_{1,2}}{c_{2,3}},\quad \frac{\left(-c'_{1,3}c'_{0,1}+2\,{c'}^2_{1,2}c'_{0,0}\right)^5}{{c'}^5_{2,3}{c'}^6_{0,1}}=\frac{\left(-c_{1,3}c_{0,1}+2\,c^2_{1,2}c_{0,0}\right)^5}{c^5_{2,3}c^6_{0,1}},$$

$$\frac{\left(c'_{2,4}c'_{0,1}-3\,c'_{2,3}c'_{0,0}c'_{1,2}+3\,{c'}^2_{2,3}c'_{0,0}\right)^5}{{c'}^5_{2,3}{c'}^6_{0,1}}=\frac{\left(c_{2,4}c_{0,1}-3\,c_{2,3}c_{0,0}c_{1,2}+3\,c^2_{2,3}c_{0,0}\right)^5}{c^5_{2,3}c^6_{0,1}}.$$

2. *For any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$, there exists $L(C)\in U_8^{12}$:*

$$\frac{c_{1,2}}{c_{2,3}}=\lambda_1,\quad \frac{\left(-c_{1,3}c_{0,1}+2\,c^2_{1,2}c_{0,0}\right)^5}{c^5_{2,3}c^6_{0,1}}=\lambda_2,\quad \frac{\left(c_{2,4}c_{0,1}-3\,c_{2,3}c_{0,0}c_{1,2}+3\,c^2_{2,3}c_{0,0}\right)^5}{c^5_{2,3}c^6_{0,1}}=\lambda_3.$$

Then orbits in U_8^{12} can be parameterized as $L\left(0,1,0,\lambda_1,\lambda_2,0,0,1,\lambda_3,0\right),\ \lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

Proof. Here, $A_1 = -\dfrac{A_0 c_{0,0}}{c_{0,1}}$, $B_1 = \dfrac{{A_0}^2}{c_{2,3}}$, $B_3 = \dfrac{1}{2\,{A_0}^2{c_{2,3}}^2{c_{0,1}}^2}({B_2}^2{c_{2,3}}^3{c_{0,1}}^2 + {A_0}^4 c_{1,4}{c_{0,1}}^2 + {A_0}^4 c_{0,0}c_{1,3}c_{2,3}c_{0,1} - 5\,{A_0}^4 c_{1,2}c_{0,0}c_{1,3}c_{0,1} + 5\,{A_0}^4{c_{1,2}}^3{c_{0,0}}^2 - {A_0}^4{c_{1,2}}^2{c_{0,0}}^2 c_{2,3})$ and
$B_4 = \dfrac{1}{6\,{A_0}^4{c_{2,3}}^3{c_{0,1}}^3}({B_2}^3{c_{2,3}}^5{c_{0,1}}^3 - 10\,{A_0}^6 c_{2,4}{c_{1,2}}^3{c_{0,0}}^2 c_{0,1} + 2\,{A_0}^6 c_{1,5}c_{2,3}{c_{0,1}}^3 - 2\,{A_0}^6 c_{2,4}c_{1,4}{c_{0,1}}^3 - 6\,{A_0}^6 c_{0,0}{c_{1,3}}^2 c_{2,3}{c_{0,1}}^2 + 17\,{A_0}^6{c_{0,0}}^3{c_{1,2}}^4 c_{2,3} + 3\,{B_2}^2{c_{2,3}}^4{A_0}^2 c_{0,0}c_{1,2}{c_{0,1}}^2 - 6\,A_3{A_0}^5 c_{1,2}{c_{2,3}}^2{c_{0,1}}^3 + 6\,B_2\,A_2{A_0}^3 c_{1,2}{c_{2,3}}^3{c_{0,1}}^3 - 21\,{A_0}^6{c_{0,0}}^3{c_{1,2}}^3{c_{2,3}}^2 - 3\,{A_0}^6 c_{1,2}c_{0,0}c_{1,4}c_{2,3}{c_{0,1}}^2 - 3\,{A_0}^6{c_{1,2}}^2{c_{0,0}}^2 c_{1,3}c_{2,3}c_{0,1} + 15\,{A_0}^6 c_{1,2}{c_{0,0}}^2 c_{1,3}{c_{2,3}}^2 c_{0,1} + 10\,{A_0}^6 c_{2,4}c_{1,2}c_{0,0}c_{1,3}{c_{0,1}}^2 + 15\,B_2{A_0}^4{c_{1,2}}^3{c_{0,0}}^2{c_{2,3}}^2 c_{0,1} + 3\,B_2{A_0}^4 c_{1,4}{c_{2,3}}^2{c_{0,1}}^3 + 3\,B_2{A_0}^4 c_{0,0}c_{1,3}{c_{2,3}}^3{c_{0,1}}^2 - 15\,B_2{A_0}^4 c_{1,2}c_{0,0}c_{1,3}{c_{2,3}}^2{c_{0,1}}^2 - 3\,B_2{A_0}^4{c_{1,2}}^2{c_{0,0}}^2{c_{2,3}}^3 c_{0,1})$. □

Proposition 2.20.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{13} are isomorphic, if and only if*
$$\frac{{c'}^7_{2,3}{c'}^{12}_{1,2}c'_{0,0}}{\left(-3\,c'_{1,3}c'_{2,3}c'_{1,2} + 3\,c'_{1,3}{c'}^2_{2,3} + 2\,{c'}^2_{1,2}c'_{2,4}\right)^6} = \frac{c^7_{2,3}c^{12}_{1,2}c_{0,0}}{\left(-3\,c_{1,3}c_{2,3}c_{1,2} + 3\,c_{1,3}c^2_{2,3} + 2\,c^2_{1,2}c_{2,4}\right)^6}, \quad \frac{c'_{1,2}}{c'_{2,3}} = \frac{c_{1,2}}{c_{2,3}}.$$
2. *For any $\lambda_1 \in \mathbb{C}$* and *$\lambda_2 \in \mathbb{C}^*$, there exists $L(C) \in U_8^{13}$:*
$$\frac{c^7_{2,3}c^{12}_{1,2}c_{0,0}}{\left(-3\,c_{1,3}c_{2,3}c_{1,2} + 3\,c_{1,3}c^2_{2,3} + 2\,c^2_{1,2}c_{2,4}\right)^6} = \lambda_1, \quad \frac{c_{1,2}}{c_{2,3}} = \lambda_2.$$

   Then orbits in U_8^{13} can be parameterized as $L(\lambda_1, 0, 0, \lambda_2, 0, 0, 0, 1, 1, 0)$, $\lambda_1 \in \mathbb{C}$ and *$\lambda_2 \in \mathbb{C}^*$.*

Proof. Here, we take $A_0 = \dfrac{-3\,c_{1,3}c_{2,3}c_{1,2} + 3\,c_{1,3}{c_{2,3}}^2 + 2\,{c_{1,2}}^2 c_{2,4}}{2\,c_{2,3}{c_{1,2}}^2}$, $A_1 = \dfrac{-A_0 c_{1,3}}{2\,{c_{1,2}}^2}$, $B_1 = \dfrac{{A_0}^2}{c_{2,3}}$,
$B_3 = \dfrac{1}{{A_0}^2{c_{2,3}}^2{c_{1,2}}^2}(4\,{B_2}^2{c_{2,3}}^3{c_{1,2}}^2 + 4\,{A_0}^4 c_{1,4}{c_{1,2}}^2 + {A_0}^4{c_{1,3}}^2 c_{2,3} - 5\,{A_0}^4{c_{1,3}}^2 c_{1,2})$, and
$B_4 = -\dfrac{1}{48\,{A_0}^4{c_{2,3}}^3{c_{1,2}}^3}(-8\,{B_2}^3{c_{2,3}}^5{c_{1,2}}^3 - 20\,{A_0}^6 c_{2,4}{c_{1,3}}^2{c_{1,2}}^2 - 16\,{A_0}^6 c_{1,5}c_{2,3}{c_{1,2}}^3 + 16\,{A_0}^6 c_{2,4}c_{1,4}{c_{1,2}}^3 + 13\,{A_0}^6{c_{1,3}}^3 c_{2,3}c_{1,2} - 12\,{B_2}^2{c_{2,3}}^4{A_0}^2 c_{1,3}{c_{1,2}}^2 + 48\,A_3{A_0}^5{c_{1,2}}^4{c_{2,3}}^2 - 48\,B_2A_2{A_0}^3{c_{1,2}}^4{c_{2,3}}^3 - 9\,{A_0}^6{c_{1,3}}^3{c_{2,3}}^2 + 12\,{A_0}^6 c_{1,3}c_{1,4}c_{2,3}{c_{1,2}}^2 + 30\,B_2{A_0}^4{c_{1,3}}^2{c_{2,3}}^2{c_{1,2}}^2 - 24\,B_2{A_0}^4 c_{1,4}{c_{2,3}}^2{c_{1,2}}^3 - 6\,B_2{A_0}^4{c_{1,3}}^2{c_{2,3}}^3 c_{1,2})$. □

Proposition 2.21.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{14} are isomorphic, if and only if* $\dfrac{c'_{1,2}}{c'_{2,3}} = \dfrac{c_{1,2}}{c_{2,3}}$.
2. *For any $\lambda \in \mathbb{C}^*$, there exists $L(C) \in U_8^{14}$:* $\dfrac{c_{1,2}}{c_{2,3}} = \lambda$.

   Then orbits in U_8^{14} can be parameterized as $L(1, 0, 0, \lambda, 0, 0, 0, 1, 0, 0)$, $\lambda \in \mathbb{C}^*$.

Proof. Here, $A_1 = -\dfrac{A_0 c_{1,3}}{2\,{c_{1,2}}^2}$, $B_1 = \dfrac{{A_0}^2}{c_{2,3}}$, $B_3 = \dfrac{4\,{B_2}^2{c_{2,3}}^3{c_{1,2}}^2 + 4\,{A_0}^4 c_{1,4}{c_{1,2}}^2 + {A_0}^4{c_{1,3}}^2 c_{2,3} - 5\,{A_0}^4{c_{1,3}}^2 c_{1,2}}{8\,{A_0}^2{c_{2,3}}^2{c_{1,2}}^2}$,
and $B_4 = \dfrac{-1}{48\,{A_0}^4{c_{2,3}}^3{c_{1,2}}^3}(-8\,{B_2}^3{c_{2,3}}^5{c_{1,2}}^3 - 20\,{A_0}^6 c_{2,4}{c_{1,3}}^2{c_{1,2}}^2 - 16\,{A_0}^6 c_{1,5}c_{2,3}{c_{1,2}}^3 + 16\,{A_0}^6 c_{2,4}c_{1,4}{c_{1,2}}^3 + 13\,{A_0}^6{c_{1,3}}^3 c_{2,3}c_{1,2} - 12\,{B_2}^2{c_{2,3}}^4{A_0}^2 c_{1,3}{c_{1,2}}^2 + 48\,A_3{A_0}^5{c_{1,2}}^4{c_{2,3}}^2 - 48\,B_2A_2{A_0}^3{c_{1,2}}^4{c_{2,3}}^3 - 9\,{A_0}^6{c_{1,3}}^3{c_{2,3}}^2 + 12\,{A_0}^6 c_{1,3}c_{1,4}c_{2,3}{c_{1,2}}^2 + 30\,B_2{A_0}^4{c_{1,3}}^2{c_{2,3}}^2{c_{1,2}}^2 - 24\,B_2{A_0}^4 c_{1,4}{c_{2,3}}^2{c_{1,2}}^3 - 6\,B_2{A_0}^4{c_{1,3}}^2{c_{2,3}}^3 c_{1,2})$. □

Proposition 2.22.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{15} are isomorphic, if and only if* $\dfrac{c'_{1,2}}{c'_{2,3}} = \dfrac{c_{1,2}}{c_{2,3}}$.
2. *For any $\lambda \in \mathbb{C}^*$, there exists $L(C) \in U_8^{15}$:* $\dfrac{c_{1,2}}{c_{2,3}} = \lambda$.

   Then orbits in U_8^{15} can be parameterized as $L(0, 0, 0, \lambda, 0, 0, 0, 1, 0, 0)$, $\lambda \in \mathbb{C}^*$.

Proof. Put, $A_1=-\dfrac{A_0c_{1,3}}{2\,c_{1,2}{}^2}$, $B_1=\dfrac{A_0{}^2}{c_{2,3}}$, $B_3=\dfrac{4\,B_2{}^2c_{2,3}{}^3c_{1,2}{}^2+4\,A_0{}^4c_{1,4}c_{1,2}{}^2+A_0{}^4c_{1,3}{}^2c_{2,3}-5\,A_0{}^4c_{1,3}{}^2c_{1,2}}{8\,A_0{}^2c_{2,3}{}^2c_{1,2}{}^2}$, and $B_4=\dfrac{-1}{48\,A_0{}^4c_{2,3}{}^3c_{1,2}{}^3}(-8\,B_2{}^3c_{2,3}{}^5c_{1,2}{}^3-20\,A_0{}^6c_{2,4}c_{1,3}{}^2c_{1,2}{}^2-16\,A_0{}^6c_{1,5}c_{2,3}c_{1,2}{}^3+16\,A_0{}^6c_{2,4}c_{1,4}c_{1,2}{}^3+13\,A_0{}^6c_{1,3}{}^3c_{2,3}c_{1,2}-12\,B_2{}^2c_{2,3}{}^4A_0{}^2c_{1,3}c_{1,2}{}^2+48\,A_3A_0{}^5c_{1,2}{}^4c_{2,3}{}^2-48\,B_2A_2A_0{}^3c_{1,2}{}^4c_{2,3}{}^3-9\,A_0{}^6c_{1,3}{}^3c_{2,3}{}^2+12\,A_0{}^6c_{1,3}c_{1,4}c_{2,3}c_{1,2}{}^2+30\,B_2A_0{}^4c_{1,3}{}^2c_{2,3}{}^2c_{1,2}{}^2-24\,B_2A_0{}^4c_{1,4}c_{2,3}{}^2c_{1,2}{}^3-6\,B_2A_0{}^4c_{1,3}{}^2c_{2,3}{}^3c_{1,2})$. □

Proposition 2.23.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{16} are isomorphic, if and only if* $\dfrac{{c'_{2,3}}^7c'_{0,0}}{{c'_{1,3}}^6}=\dfrac{c_{2,3}{}^7c_{0,0}}{c_{1,3}{}^6}$.

2. *For any $\lambda_1\in\mathbb{C}$, there exists $L(C)\in U_8^{16}$:* $\dfrac{c_{2,3}{}^7c_{0,0}}{c_{1,3}{}^6}=\lambda_1$.

   Then orbits in U_8^{16} can be parameterized as $L\left(\lambda_1,0,0,0,1,0,0,1,0,0\right),\ \lambda_1\in\mathbb{C}$.

Proof. For this case we put $A_0=\dfrac{c_{1,3}}{c_{2,3}}$, $A_1=\dfrac{c_{1,3}c_{2,4}}{3\,c_{2,3}{}^3}$, $B_1=\dfrac{c_{1,3}{}^2}{c_{2,3}{}^3}$, $B_3=\dfrac{3\,B_2{}^2c_{2,3}{}^8+3\,c_{1,3}{}^4c_{1,4}c_{2,3}-c_{1,3}{}^5c_{2,4}}{6\,c_{1,3}{}^2c_{2,3}{}^5}$, and $B_4=\dfrac{1}{6\,c_{1,3}{}^4c_{2,3}{}^6}(B_2{}^3c_{2,3}{}^{12}+2\,c_{1,3}{}^6c_{1,5}c_{2,3}{}^2-2\,c_{1,3}{}^6c_{2,4}c_{1,4}c_{2,3}+2\,c_{1,3}{}^8c_{2,4}+3\,B_2c_{1,3}{}^4c_{1,4}c_{2,3}{}^5-B_2c_{1,3}{}^5c_{2,4}c_{2,3}{}^4)$. □

Proposition 2.24.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{19} are isomorphic, if and only if*

$$\frac{\left(4\,c'_{0,0}{c'}^4_{1,2}-2\,c'_{1,3}c'_{0,1}{c'}^2_{1,2}+{c'}^2_{1,3}c'_{1,1}\right){c'}^3_{1,2}}{{c'}^6_{2,4}}=\frac{\left(4\,c_{0,0}c^4_{1,2}-2\,c_{1,3}c_{0,1}c^2_{1,2}+c^2_{1,3}c_{1,1}\right)c^3_{1,2}}{c^6_{2,4}},$$

$$\frac{\left(-c'_{1,3}c'_{1,1}+c'_{0,1}{c'}^2_{1,2}\right){c'}^3_{1,2}}{{c'}^5_{2,4}}=\frac{\left(-c_{1,3}c_{1,1}+c_{0,1}c^2_{1,2}\right)c^3_{1,2}}{c^5_{2,4}},\quad \frac{{c'}^3_{1,2}c'_{1,1}}{{c'}^4_{2,4}}=\frac{c^3_{1,2}c_{1,1}}{c^4_{2,4}},$$

$$\frac{4\,c'_{1,4}c'_{1,2}-5\,{c'}^2_{1,3}}{{c'}^2_{2,4}}=\frac{4\,c_{1,4}c_{1,2}-5\,c^2_{1,3}}{c^2_{2,4}}.$$

2. *For any $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{C}$, there exists $L(C)\in U_8^{19}$:* $\dfrac{\left(4\,c_{0,0}c^4_{1,2}-2\,c_{1,3}c_{0,1}c^2_{1,2}+c^2_{1,3}c_{1,1}\right)c^3_{1,2}}{c^6_{2,4}}=\lambda_1,$

$$\frac{\left(-c_{1,3}c_{1,1}+c_{0,1}c^2_{1,2}\right)c^3_{1,2}}{c^5_{2,4}}=\lambda_2,\quad \frac{c^3_{1,2}c_{1,1}}{c^4_{2,4}}=\lambda_3,\quad \frac{4\,c_{1,4}c_{1,2}-5\,c^2_{1,3}}{c^2_{2,4}}=\lambda_4.$$

   Then orbits in U_8^{19} can be parameterized as $L\left(\lambda_1,\lambda_2,\lambda_3,1,0,\lambda_4,0,0,1,0\right),\quad \lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{C}$.

Proof. Put $A_0=\dfrac{c_{2,4}}{c_{1,2}}$, $A_1=-\dfrac{c_{1,3}c_{2,4}}{2\,c_{1,2}{}^3}$, $B_1=\dfrac{c_{2,4}{}^2}{c_{1,2}{}^3}$, and

$$B_3=\frac{1}{8\,c_{1,2}{}^5c_{2,4}{}^2}(8\,c_{2,4}{}^3c_{1,3}{}^3+4\,B_2{}^2c_{1,2}{}^8-12\,c_{2,4}{}^3c_{1,3}c_{1,4}c_{1,2}+4\,c_{2,4}{}^3c_{1,5}c_{1,2}{}^2+c_{2,4}{}^4c_{1,3}{}^2).$$

□

Proposition 2.25.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{20} are isomorphic, if and only if*

$$\frac{\left(4\,c'_{0,0}c'_{1,1}-{c'}^2_{0,1}\right)^3{c'}^{10}_{2,4}}{{c'}^{10}_{1,1}}=\frac{\left(4\,c_{0,0}c_{1,1}-c^2_{0,1}\right)^3c^{10}_{2,4}}{c^{10}_{1,1}},\quad\frac{c'_{1,3}}{c'_{2,4}}=\frac{c_{1,3}}{c_{2,4}},\quad\frac{{c'}^3_{1,4}}{{c'}^2_{2,4}c'_{1,1}}=\frac{c^3_{1,4}}{c^2_{2,4}c_{1,1}}.$$

2. *For any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$, there exists $L(C)\in U_8^{20}$:*

$$\frac{\left(4\,c_{0,0}c_{1,1}-c^2_{0,1}\right)^3c^{10}_{2,4}}{c^{10}_{1,1}}=\lambda_1,\quad\frac{c_{1,3}}{c_{2,4}}=\lambda_2,\quad\frac{c^3_{1,4}}{c^2_{2,4}c_{1,1}}=\lambda_3.$$

Then orbits in U_8^{20} can be parameterized as $L\left(\lambda_1,0,1,0,\lambda_2,\lambda_3,0,0,1,0\right),\ \lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

Proof. Here,

$$A_1=-\frac{A_0c_{0,1}}{2\,c_{1,1}},\quad B_1=\frac{{A_0}^3}{c_{2,4}},\ \text{and}\ B_3=\frac{2\,{B_2}^2{c_{2,4}}^3c_{1,1}-3\,{A_0}^6c_{0,1}{c_{1,3}}^2+2\,{A_0}^6c_{1,5}c_{1,1}+{A_0}^6c_{0,1}c_{1,3}c_{2,4}}{4\,{A_0}^3{c_{2,4}}^2c_{1,1}}.$$

□

Proposition 2.26.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{21} are isomorphic, if and only if*

$$\frac{c'_{1,3}}{c'_{2,4}}=\frac{c_{1,3}}{c_{2,4}},\quad\frac{{c'}^5_{1,4}}{{c'}^5_{2,4}c'_{0,1}}=\frac{c^5_{1,4}}{c^5_{2,4}c_{0,1}}.$$

2. *For any $\lambda_1,\lambda_2\in\mathbb{C}$, there exists $L(C)\in U_8^{21}$: $\dfrac{c_{1,3}}{c_{2,4}}=\lambda_1,\quad\dfrac{c^5_{1,4}}{c^5_{2,4}c_{0,1}}=\lambda_2$.*

Then orbits in U_8^{21} can be parameterized as $L\left(0,0,1,0,0,\lambda_1,\lambda_2,0,0,1,0\right),\ \lambda_1,\lambda_2\in\mathbb{C}$.

Proof. We put,

$$A_1=-\frac{A_0c_{0,0}}{c_{0,1}},\quad B_1=\frac{{A_0}^3}{c_{2,4}},\ \text{and}\ B_3=\frac{{B_2}^2{c_{2,4}}^3c_{0,1}-3\,{A_0}^6c_{0,0}{c_{1,3}}^2+{A_0}^6c_{1,5}c_{0,1}+{A_0}^6c_{0,0}c_{1,3}c_{2,4}}{2\,{A_0}^3{c_{2,4}}^2c_{0,1}}.$$

□

Proposition 2.27.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{22} are isomorphic, if and only if*

$$\frac{{c'}^8_{2,4}c'_{0,0}}{{c'}^7_{1,4}}=\frac{c^8_{2,4}c_{0,0}}{c^7_{1,4}},\quad\frac{c'_{1,3}}{c'_{2,4}}=\frac{c_{1,3}}{c_{2,4}}.$$

2. *For any $\lambda_1,\lambda_2\in\mathbb{C}$ there exists $L(C)\in U_8^{22}$: $\dfrac{c^8_{2,4}c_{0,0}}{c^7_{1,4}}=\lambda_1,\ \dfrac{c_{1,3}}{c_{2,4}}=\lambda_2$.*

Then orbits in U_8^{22} can be parameterized as $L\left(\lambda_1,0,0,0,\lambda_2,1,0,0,1,0\right),\ \lambda_1,\lambda_2\in\mathbb{C}$.

Proof. We put here,

$$A_0=\frac{c_{1,4}}{c_{2,4}},\quad B_1=\frac{{c_{1,4}}^3}{{c_{2,4}}^4},\ \text{and}\ B_3=\frac{{B_2}^2{c_{2,4}}^9+3\,{c_{1,4}}^5A_1{c_{1,3}}^2c_{2,4}+{c_{1,4}}^6c_{1,5}-{c_{1,4}}^5A_1c_{1,3}{c_{2,4}}^2}{2\,{c_{2,4}}^5{c_{1,4}}^3}.$$

□

Proposition 2.28.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{23} *are isomorphic, if and only if* $\dfrac{c'_{1,3}}{c'_{2,4}} = \dfrac{c_{1,3}}{c_{2,4}}$.

2. *For any* $\lambda \in \mathbb{C}^*$, *there exists* $L(C) \in U_8^{23}$: $\dfrac{c_{1,3}}{c_{2,4}} = \lambda$.

   Then orbits in U_8^{23} *can be parameterized as* $L(1,0,0,0,\lambda,0,0,0,1,0)$, $\lambda \in \mathbb{C}^*$.

Proof. Put

$$B_1 = \frac{{A_0}^3}{c_{2,4}}, \text{ and } B_3 = \frac{3\,{A_0}^5 A_1 {c_{1,3}}^2 + {B_2}^2 {c_{2,4}}^3 + {A_0}^6 c_{1,5} - {A_0}^5 A_1 c_{1,3} c_{2,4}}{2\,{A_0}^3 {c_{2,4}}^2}.$$

□

Proposition 2.29.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{24} *are isomorphic, if and only if* $\dfrac{c'_{1,3}}{c'_{2,4}} = \dfrac{c_{1,3}}{c_{2,4}}$.

2. *For any* $\lambda \in \mathbb{C}^*$, *there exists* $L(C) \in U_8^{24}$: $\dfrac{c_{1,3}}{c_{2,4}} = \lambda$.

   Then orbits in U_8^{24} *can be parameterized as* $L(0,0,0,0,\lambda,0,0,0,1,0)$, $\lambda \in \mathbb{C}^*$.

Proof. Here,

$$B_1 = \frac{{A_0}^3}{c_{2,4}}, \text{ and } B_3 = \frac{3\,{A_0}^5 A_1 {c_{1,3}}^2 + {B_2}^2 {c_{2,4}}^3 + {A_0}^6 c_{1,5} - {A_0}^5 A_1 c_{1,3} c_{2,4}}{2\,{A_0}^3 {c_{2,4}}^2}.$$

□

Proposition 2.30.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{25} *are isomorphic, if and only if*

$$\frac{\left(4\,c'_{0,0}{c'}^4_{1,2} - 2\,c'_{1,3}c'_{0,1}{c'}^2_{1,2} + {c'}^2_{1,3}{c'}_{1,1}\right){c'}^3_{1,2}}{\left(4\,c'_{1,4}c'_{1,2} - 5\,{c'}^2_{1,3}\right)^3} = \frac{\left(4\,c_{0,0}c^4_{1,2} - 2\,c_{1,3}c_{0,1}c^2_{1,2} + c^2_{1,3}c_{1,1}\right)c^3_{1,2}}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^3},$$

$$\frac{\left(-c'_{1,3}c'_{1,1} + c'_{0,1}{c'}^2_{1,2}\right)^2 {c'}^6_{1,2}}{\left(4\,c'_{1,4}c'_{1,2} - 5\,{c'}^2_{1,3}\right)^5} = \frac{\left(-c_{1,3}c_{1,1} + c_{0,1}c^2_{1,2}\right)^2 c^6_{1,2}}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^5},$$

$$\frac{{c'}^3_{1,2}c'_{1,1}}{\left(4\,c'_{1,4}c'_{1,2} - 5\,{c'}^2_{1,3}\right)^2} = \frac{c^3_{1,2}c_{1,1}}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^2},$$

$$\frac{\left(2\,{c'}^3_{1,3} + c'_{1,5}{c'}^2_{1,2} - 3\,c'_{1,3}c'_{1,4}c'_{1,2}\right)^2}{\left(4\,c'_{1,4}c'_{1,2} - 5\,{c'}^2_{1,3}\right)^3} = \frac{\left(2\,c^3_{1,3} + c_{1,5}c^2_{1,2} - 3\,c_{1,3}c_{1,4}c_{1,2}\right)^2}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^3}.$$

2. *For any* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$, *there exists* $L(C) \in U_8^{25}$:

$$\frac{\left(4\,c_{0,0}c^4_{1,2} - 2\,c_{1,3}c_{0,1}c^2_{1,2} + c^2_{1,3}c_{1,1}\right)c^3_{1,2}}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^3} = \lambda_1, \quad \frac{\left(-c_{1,3}c_{1,1} + c_{0,1}c^2_{1,2}\right)^2 c^6_{1,2}}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^5} = \lambda_2,$$

$$\frac{c^3_{1,2}c_{1,1}}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^2} = \lambda_3, \qquad \frac{\left(2\,c^3_{1,3} + c_{1,5}c^2_{1,2} - 3\,c_{1,3}c_{1,4}c_{1,2}\right)^2}{\left(4\,c_{1,4}c_{1,2} - 5\,c^2_{1,3}\right)^3} = \lambda_4.$$

   Then orbits in U_8^{25} *can be parameterized as* $L(\lambda_1,\lambda_2,\lambda_3,1,0,1,\lambda_4,0,0,0)$, $\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in \mathbb{C}$.

Proof. Here,

$$A_1 = -\frac{A_0 c_{1,3}}{2\, {c_{1,2}}^2}, \text{ and } B_1 = \frac{{A_0}^2}{c_{1,2}}.$$

□

Proposition 2.31.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{26} are isomorphic, if and only if*

$$\frac{\left(4\, c'_{0,0}{c'}^4_{1,2} - 2\, c'_{1,3}c'_{0,1}{c'}^2_{1,2} + {c'}^2_{1,3}{c'}_{1,1}\right){c'}^3_{1,2}}{\left(-7\, {c'}^3_{1,3} + 4\, {c'}_{1,5}{c'}^2_{1,2}\right)^2} = \frac{\left(4\, c_{0,0}c^4_{1,2} - 2\, c_{1,3}c_{0,1}c^2_{1,2} + c^2_{1,3}c_{1,1}\right)c^3_{1,2}}{\left(-7\, c^3_{1,3} + 4\, c_{1,5}c^2_{1,2}\right)^2},$$

$$\frac{\left(-c'_{1,3}c'_{1,1} + c'_{0,1}{c'}^2_{1,2}\right)^3 {c'}^9_{1,2}}{\left(-7\, {c'}^3_{1,3} + 4\, c'_{1,5}{c'}^2_{1,2}\right)^5} = \frac{\left(-c_{1,3}c_{1,1} + c_{0,1}c^2_{1,2}\right)^3 c^9_{1,2}}{\left(-7\, c^3_{1,3} + 4\, c_{1,5}c^2_{1,2}\right)^5},$$

$$\frac{{c'}^9_{1,2}{c'}^3_{1,1}}{\left(-7\, {c'}^3_{1,3} + 4\, c'_{1,5}{c'}^2_{1,2}\right)^4} = \frac{c^9_{1,2}c^3_{1,1}}{\left(-7\, c^3_{1,3} + 4\, c_{1,5}c^2_{1,2}\right)^4}.$$

2. *For any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, there exists $L(C) \in U_8^{26}$:* $\dfrac{\left(4\, c_{0,0}c^4_{1,2} - 2\, c_{1,3}c_{0,1}c^2_{1,2} + c^2_{1,3}c_{1,1}\right)c^3_{1,2}}{\left(-7\, c^3_{1,3} + 4\, c_{1,5}c^2_{1,2}\right)^2} = \lambda_1,$

$$\frac{\left(-c_{1,3}c_{1,1} + c_{0,1}c^2_{1,2}\right)^3 c^9_{1,2}}{\left(-7\, c^3_{1,3} + 4\, c_{1,5}c^2_{1,2}\right)^5} = \lambda_2, \quad \frac{c^9_{1,2}c^3_{1,1}}{\left(-7\, c^3_{1,3} + 4\, c_{1,5}c^2_{1,2}\right)^4} = \lambda_3.$$

Then orbits in U_8^{26} can be parameterized as $L\left(\lambda_1, \lambda_2, \lambda_3, 1, 0, 0, 1, 0, 0, 0\right)$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

Proof. For this case, put in the base change (9),

$$A_1 = -\frac{A_0 c_{1,3}}{2\, {c_{1,2}}^2}, \text{ and } B_1 = \frac{{A_0}^2}{c_{1,2}}.$$

□

Proposition 2.32.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{27} are isomorphic, if and only if*

$$\frac{\left(4\, c'_{0,0}{c'}^4_{1,2} - 2\, c'_{1,3}c'_{0,1}{c'}^2_{1,2} + {c'}^2_{1,3}c'_{1,1}\right)^2}{{c'}^3_{1,2}{c'}^3_{1,1}} = \frac{\left(4\, c_{0,0}c^4_{1,2} - 2\, c_{1,3}c_{0,1}c^2_{1,2} + c^2_{1,3}c_{1,1}\right)^2}{c^3_{1,2}c^3_{1,1}},$$

$$\frac{\left(-c'_{1,3}c'_{1,1} + c'_{0,1}{c'}^2_{1,2}\right)^4}{{c'}^3_{1,2}{c'}^5_{1,1}} = \frac{\left(-c_{1,3}c_{1,1} + c_{0,1}c^2_{1,2}\right)^4}{c^3_{1,2}c^5_{1,1}}.$$

2. *For any $\lambda_1, \lambda_2 \in \mathbb{C}$, there exists $L(C) \in U_8^{27}$:*

$$\frac{\left(4\, c_{0,0}c^4_{1,2} - 2\, c_{1,3}c_{0,1}c^2_{1,2} + c^2_{1,3}c_{1,1}\right)^2}{c^3_{1,2}c^3_{1,1}} = \lambda_1, \quad \frac{\left(-c_{1,3}c_{1,1} + c_{0,1}c^2_{1,2}\right)^4}{c^3_{1,2}c^5_{1,1}} = \lambda_2.$$

Then orbits in U_8^{27} can be parameterized as $L\left(\lambda_1, \lambda_2, 1, 1, 0, 0, 0, 0, 0, 0\right)$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. Put in the base change (9),

$$A_1 = -\frac{A_0 c_{1,3}}{2\, {c_{1,2}}^2}, \text{ and } B_1 = \frac{{A_0}^2}{c_{1,2}}.$$

□

Proposition 2.33.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{28} are isomorphic, if and only if*

$$\frac{\left(2\, c'_{0,0} {c'}^2_{1,2} - c'_{1,3} c'_{0,1}\right)^5}{{c'}^5_{1,2} {c'}^6_{0,1}} = \frac{\left(2\, c_{0,0} c^2_{1,2} - c_{1,3} c_{0,1}\right)^5}{c^5_{1,2} c^6_{0,1}}.$$

2. *For any $\lambda_1 \in \mathbb{C}$, there exists $L(C) \in U_8^{28}$:* $\dfrac{\left(2\, c_{0,0} c^2_{1,2} - c_{1,3} c_{0,1}\right)^5}{c^5_{1,2} c^6_{0,1}} = \lambda_1.$

   Then orbits in U_8^{28} can be parameterized as $L(\lambda_1, 1, 0, 1, 0, 0, 0, 0, 0, 0)$, $\lambda_1 \in \mathbb{C}$.

Proof. For this case, we put

$$A_1 = -\frac{A_0 c_{1,3}}{2\, {c_{1,2}}^2}, \text{ and } B_1 = \frac{{A_0}^2}{c_{1,2}}.$$

□

Proposition 2.34.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{31} are isomorphic, if and only if*

$$\frac{{c'}^4_{1,3}\left(-9\, c'_{0,0} {c'}^4_{1,3} + 3\, c'_{1,5} c'_{0,1} {c'}^2_{1,3} - {c'}^2_{1,5} c'_{1,1}\right)}{{c'}^7_{1,4}} = \frac{c^4_{1,3}\left(-9\, c_{0,0} c^4_{1,3} + 3\, c_{1,5} c_{0,1} c^2_{1,3} - c^2_{1,5} c_{1,1}\right)}{c^7_{1,4}},$$

$$\frac{{c'}^3_{1,3}\left(-2\, c'_{1,5} c'_{1,1} + 3\, c'_{0,1} {c'}^2_{1,3}\right)}{{c'}^5_{1,4}} = \frac{c^3_{1,3}\left(-2\, c_{1,5} c_{1,1} + 3\, c_{0,1} c^2_{1,3}\right)}{c^5_{1,4}}, \quad \frac{{c'}^2_{1,3} c'_{1,1}}{{c'}^3_{1,4}} = \frac{c^2_{1,3} c_{1,1}}{c^3_{1,4}}.$$

2. *For any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, there exists $L(C) \in U_8^{31}$:* $\dfrac{c^4_{1,3}\left(-9\, c_{0,0} c^4_{1,3} + 3\, c_{1,5} c_{0,1} c^2_{1,3} - c^2_{1,5} c_{1,1}\right)}{c^7_{1,4}} = \lambda_1,$

   $\dfrac{c^3_{1,3}\left(-2\, c_{1,5} c_{1,1} + 3\, c_{0,1} c^2_{1,3}\right)}{c^5_{1,4}} = \lambda_2, \quad \dfrac{c^2_{1,3} c_{1,1}}{c^3_{1,4}} = \lambda_3.$

   Then orbits in U_8^{31} can be parameterized as $L(\lambda_1, \lambda_2, \lambda_3, 0, 1, 1, 0, 0, 0, 0)$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

Proof. Here, we put in the base change (9) the following coefficients:

$$A_0 = \frac{c_{1,4}}{c_{1,3}}, \quad A_1 = -\frac{c_{1,4} c_{1,5}}{3\, {c_{1,3}}^3}, \text{ and } B_1 = \frac{{c_{1,4}}^3}{{c_{1,3}}^4}.$$

□

Proposition 2.35.

1. *Two algebras $L(C)$ and $L(C')$ from U_8^{32} are isomorphic, if and only if*

$$\frac{\left(-9\, c'_{0,0} {c'}^4_{1,3} + 3\, c'_{1,5} c'_{0,1} {c'}^2_{1,3} - {c'}^2_{1,5} c'_{1,1}\right)^3}{{c'}^2_{1,3} {c'}^7_{1,1}} = \frac{\left(-9\, c_{0,0} c^4_{1,3} + 3\, c_{1,5} c_{0,1} c^2_{1,3} - c^2_{1,5} c_{1,1}\right)^3}{c^2_{1,3} c^7_{1,1}},$$

$$\frac{\left(-2\, c'_{1,5} c'_{1,1} + 3\, c'_{0,1} {c'}^2_{1,3}\right)^3}{c'_{1,3} {c'}^5_{1,1}} = \frac{\left(-2\, c_{1,5} c_{1,1} + 3\, c_{0,1} c^2_{1,3}\right)^3}{c_{1,3} c^5_{1,1}}.$$

2. *For any* $\lambda_1, \lambda_2 \in \mathbb{C}$*, there exists* $L(C) \in U_8^{32}$:

$$\frac{\left(-9\, c_{0,0}c_{1,3}^4 + 3\, c_{1,5}c_{0,1}c_{1,3}^2 - c_{1,5}^2 c_{1,1}\right)^3}{c_{1,3}^2 c_{1,1}^7} = \lambda_1, \; \frac{\left(-2\, c_{1,5}c_{1,1} + 3\, c_{0,1}c_{1,3}^2\right)^3}{c_{1,3}c_{1,1}^5} = \lambda_2.$$

Then orbits in U_8^{32} *can be parameterized as* $L(\lambda_1, \lambda_2, 1, 0, 1, 0, 0, 0, 0, 0),\ \lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. We put

$$A_1 = -\frac{A_0 c_{1,5}}{3\, {c_{1,3}}^2}, \text{ and } B_1 = \frac{{A_0}^3}{c_{1,3}}.$$

□

Proposition 2.36.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{33} *are isomorphic, if and only if*

$$\frac{\left(3\, c'_{0,0}{c'}_{1,3}^2 - c'_{1,5}c'_{0,1}\right)^5}{{c'}_{1,3}^5 {c'}_{0,1}^7} = \frac{\left(3\, c_{0,0}c_{1,3}^2 - c_{1,5}c_{0,1}\right)^5}{c_{1,3}^5 c_{0,1}^7}.$$

2. *For any* $\lambda_1 \in \mathbb{C}$*, there exists* $L(C) \in U_8^{33}$: $\dfrac{\left(3\, c_{0,0}c_{1,3}^2 - c_{1,5}c_{0,1}\right)^5}{c_{1,3}^5 c_{0,1}^7} = \lambda_1.$

   Then orbits in U_8^{33} *can be parameterized as* $L(\lambda_1, 1, 0, 0, 1, 0, 0, 0, 0, 0), \quad \lambda_1 \in \mathbb{C}$.

Proof. Here,

$$A_1 = -\frac{A_0 c_{1,5}}{3\, {c_{1,3}}^2}, \text{ and } B_1 = \frac{{A_0}^3}{c_{1,3}}.$$

□

Proposition 2.37.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{36} *are isomorphic, if and only if*

$$\frac{{c'}_{1,5}^{10}\left(4\, {c'}_{0,0}c'_{1,1} - {c'}_{0,1}^2\right)}{{c'}_{1,1}^{10}} = \frac{c_{1,5}^{10}\left(4\, c_{0,0}c_{1,1} - c_{0,1}^2\right)}{c_{1,1}^{10}}, \quad \frac{c'_{1,1}c'_{1,4}}{{c'}_{1,5}^2} = \frac{c_{1,1}c_{1,4}}{c_{1,5}^2}.$$

2. *For any* $\lambda_1, \lambda_2 \in \mathbb{C}$*, there exists* $L(C) \in U_8^{36}$: $\dfrac{c_{1,5}^{10}\left(4\, c_{0,0}c_{1,1} - c_{0,1}^2\right)}{c_{1,1}^{10}} = \lambda_1, \quad \dfrac{c_{1,1}c_{1,4}}{c_{1,5}^2} = \lambda_2.$

   Then orbits in U_8^{36} *can be parameterized as* $L(\lambda_1, 0, 1, 0, 0, \lambda_2, 1, 0, 0, 0), \quad \lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. Her, we take

$$A_0 = \frac{c_{1,1}}{c_{1,5}}, \; A_1 = -\frac{c_{0,1}}{2\, c_{1,5}}, \text{ and } B_1 = \frac{{c_{1,1}}^5}{{c_{1,5}}^6}.$$

□

Proposition 2.38.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{37} *are isomorphic, if and only if*

$$\frac{c_{1,4}^5\left(4\, c_{0,0}c_{1,1} - c_{0,1}^2\right)}{c_{1,1}^5} = \frac{c_{1,4}^5\left(4\, c_{0,0}c_{1,1} - c_{0,1}^2\right)}{c_{1,1}^5}.$$

2. *For any* $\lambda_1 \in \mathbb{C}$, *there exists* $L(C) \in U_8^{37}$: $\dfrac{c_{1,4}^5\left(4\,c_{0,0}c_{1,1} - c_{0,1}^2\right)}{c_{1,1}^5} = \lambda_1$.

   Then orbits in U_8^{37} *can be parameterized as* $L\left(\lambda_1, 0, 1, 0, 0, 1, 0, 0, 0, 0\right), \quad \lambda_1 \in \mathbb{C}$.

Proof. We put in the base change (9),

$$A_0 = \sqrt{\frac{c_{1,1}}{c_{1,4}}},\ A_1 = -\frac{A_0 c_{0,1}}{2\,c_{1,1}},\ \text{and}\ B_1 = \frac{{c_{1,1}}^5}{{c_{1,5}}^6}.$$

□

Proposition 2.39.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{40} *are isomorphic, if and only if* $\dfrac{c_{0,1}c_{1,4}^5}{c_{1,5}^5} = \dfrac{c_{0,1}c_{1,4}^5}{c_{1,5}^5}$.

2. *For any* $\lambda_1 \in \mathbb{C}$ *there exists* $L(C) \in U_8^{40}$: $\dfrac{c_{0,1}c_{1,4}^5}{c_{1,5}^5} = \lambda_1$.

   Then orbits in U_8^{40} *can be parameterized as* $L\left(0, 1, 0, 0, 0, \lambda_1, 1, 0, 0, 0\right), \quad \lambda_1 \in \mathbb{C}$.

Proof. Here,

$$A_1 = -\frac{A_0 c_{0,0}}{c_{0,1}},\ \text{and}\ B_1 = \frac{c_{0,1}}{c_{1,5}}.$$

□

Proposition 2.40.

1. *Two algebras* $L(C)$ *and* $L(C')$ *from* U_8^{43} *are isomorphic, if and only if* $\dfrac{c_{1,4}^9 c_{0,0}}{c_{1,5}^8} = \dfrac{c_{1,4}^9 c_{0,0}}{c_{1,5}^8}$.

2. *For any* $\lambda_1 \in \mathbb{C}$ *there exists* $L(C) \in U_8^{43}$: $\dfrac{c_{1,4}^9 c_{0,0}}{c_{1,5}^8} = \lambda_1$.

   Then orbits in U_8^{43} *can be parameterized as* $L\left(\lambda_1, 0, 0, 0, 0, 1, 1, 0, 0, 0\right), \quad \lambda_1 \in \mathbb{C}$.

Proof. Finally, put

$$A_0 = \frac{c_{1,5}}{c_{1,4}},\ \text{and}\ B_1 = \frac{{c_{1,5}}^4}{{c_{1,4}}^5}.$$

□

Proposition 2.41.
The subsets U_8^5, U_8^6, U_8^7, U_8^8, U_8^9, U_8^{10}, U_8^{17}, U_8^{18}, U_8^{29}, U_8^{30}, U_8^{34}, U_8^{35}, U_8^{38}, U_8^{39}, U_8^{41}, U_8^{42}, U_8^{44}, U_8^{45}, U_8^{46}, U_8^{47}, U_8^{48} *and* U_8^{49} *are single orbits with representatives* $L(1,0,0,0,1,0,0,0,0,1)$, $L(0,0,0,0,1,0,0,0,0,1)$, $L(1,0,0,0,0,1,0,0,0,1)$, $L(0,0,0,0,0,1,0,0,0,1)$, $L(1,0,0,0,0,0,0,0,0,1)$, $L(0,0,0,0,0,0,0,0,0,1)$, $L(1,0,0,0,0,0,0,1,0,0)$, $L(0,0,0,0,0,0,0,1,0,0)$, $L(1,0,0,1,0,0,0,0,0,0)$, $L(0,0,0,1,0,0,0,0,0,0)$, $L(1,0,0,0,1,0,0,0,0,0)$, $L(0,0,0,0,1,0,0,0,0,0)$, $L(1,0,1,0,0,0,0,0,0,0)$, $L(0,0,1,0,0,0,0,0,0,0)$, $L(0,1,0,0,0,1,0,0,0,0)$, $L(0,1,0,0,0,0,0,0,0,0)$, $L(1,0,0,0,0,1,0,0,0,0)$, $L(0,0,0,0,0,1,0,0,0,0)$, $L(1,0,0,0,0,0,1,0,0,0)$, $L(0,0,0,0,0,0,1,0,0,0)$, $L(1,0,0,0,0,0,0,0,0,0)$ *and* $L(0,0,0,0,0,0,0,0,0,0)$, *respectively.*

Proof. To prove it, we give the appropriate values of $A_0, A_1, A_2, A_3, B_1, B_2, B_3$ and B_4 in the base change (9)(as for other $A_i,\ i = 4, ..., 7$, and $B_j,\ j = 5, 6, 7$ they are any, except where specified otherwise).
For U_8^5 and U_8^6 :
$B_1 = \dfrac{A_0 + A_1 c_{3,4}}{c_{3,4}}$, and $B_3 = \dfrac{-1}{2\,A_0 c_{3,4}{}^2\left(A_0 + A_1 c_{3,4}\right)}({A_0}^3 c_{2,4} + 2\,{A_0}^2 c_{2,4} A_1 c_{3,4} + A_0 c_{2,4} {A_1}^2 c_{3,4}{}^2 - A_1 c_{1,3} {A_0}^2 c_{3,4} - 2\,{A_1}^2 c_{1,3} A_0 c_{3,4}{}^2 - {A_1}^3 c_{1,3} c_{3,4}{}^3 - A_0 {B_2}^2 c_{3,4}{}^3)$.

For U_8^7 and U_8^8 :
$B_1 = \dfrac{A_0 + A_1 c_{3,4}}{c_{3,4}}$ and $B_3 = \dfrac{-1}{2\, {c_{3,4}}^2 \left(A_0 + A_1 c_{3,4}\right)} ({A_0}^2 c_{2,4} + 2\, A_0 c_{2,4} A_1 c_{3,4} + c_{2,4} {A_1}^2 {c_{3,4}}^2 - {B_2}^2 {c_{3,4}}^3)$.
For U_8^9 and U_8^{10} :

$$B_1 = \frac{A_0 + A_1 c_{3,4}}{c_{3,4}} \text{ and } B_3 = -\frac{{A_0}^2 c_{2,4} + 2\, A_0 c_{2,4} A_1 c_{3,4} + c_{2,4} {A_1}^2 {c_{3,4}}^2 - {B_2}^2 {c_{3,4}}^3}{2\, {c_{3,4}}^2 \left(A_0 + A_1 c_{3,4}\right)}.$$

For U_8^{17} and U_8^{18} :

$$A_1 = \frac{A_0 c_{2,4}}{3\, {c_{2,3}}^2}, \quad B_1 = \frac{{A_0}^2}{c_{2,3}}, \quad B_3 = \frac{{B_2}^2 {c_{2,3}}^3 + {A_0}^4 c_{1,4}}{2\, {A_0}^2 {c_{2,3}}^2}, \text{ and}$$

$$B_4 = -\frac{-{B_2}^3 {c_{2,3}}^5 - 2\, {A_0}^6 c_{1,5} c_{2,3} + 2\, {A_0}^6 c_{2,4} c_{1,4} - 3\, B_2 {A_0}^4 c_{1,4} {c_{2,3}}^2}{6\, {A_0}^4 {c_{2,3}}^3}.$$

For U_8^{29} and U_8^{30} :

$$A_1 = -\frac{A_0 c_{1,3}}{2\, {c_{1,2}}^2}, \text{ and } B_1 = \frac{{A_0}^2}{c_{1,2}}.$$

For U_8^{34} and U_8^{35} :

$$A_1 = -\frac{A_0 c_{1,5}}{3\, {c_{1,3}}^2}, \text{ and } B_1 = \frac{{A_0}^3}{c_{1,3}}.$$

For U_8^{38} and U_8^{39} :

$$A_1 = -\frac{A_0 c_{0,1}}{2\, c_{1,1}}, \text{ and } B_1 = \frac{{A_0}^6}{c_{1,1}}.$$

For U_8^{41} and U_8^{42} :

$$A_1 = -\frac{A_0 c_{0,0}}{c_{0,1}}.$$

For U_8^{44} and U_8^{45} :

$$B_1 = \frac{{A_0}^4}{c_{1,4}}.$$

For U_8^{46} and U_8^{47} :

$$B_1 = \frac{{A_0}^5}{c_{1,5}}.$$

$\square$

3 Conclusion

1. In $TLeib_7$ we distinguished 26 isomorphism classes (12 parametric family and 14 concrete) of seven dimensional Leibniz algebras and shown that they exhaust all possible cases.

2. In the case of $TLeib_8$ there are 49 isomorphism classes (27 parametric family and 22 concrete) and they exhaust all possible cases.

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